How often do conics have points?

Varieties with many rational points
LMSI Mumbai April 2026

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A brief history of a conjecture

In 1989, Y.I. Manin asked J.-P. Serre the following question:

> How many integer triples (a, b, c) are there such that $|a|, |b|, |c| \leqslant B$ and such that the equation
>
> $$aX^2 + bY^2 + cZ^2 = 0,$$
>
> has a non-trivial solution?

Equivalently, in the family $X \xrightarrow{\pi} \mathbb{P}^2$ of diagonal planar conics, how often does a fibre have a $\mathbb{Q}$-point?
Serre considered the more general setting where $X \xrightarrow{\pi} \mathbb{P}^n$ is a family of conics.

Serre's 1990 paper

Let $X \xrightarrow{\pi} \mathbb{P}^n$ be a dominant morphism from a smooth projective variety such that each fibre is a conic. For a real parameter B, define the counting function

$$N(\pi, B) = \#\{x \in \mathbb{P}^n(\mathbb{Q}) : H(x) \leqslant B \text{ and } \pi^{-1}(x)(\mathbb{Q}) \neq \varnothing\}.$$

(H is an anticanonical height, so $\#\{x \in \mathbb{P}^n(\mathbb{Q}) : H(x) \leqslant B\} \sim cB$.)

Serre introduced an invariant $\Delta(\pi) \geqslant 0$ of the fibration and then used the large sieve to show that

$$N(\pi, B) \ll \frac{B}{(\log B)^{\Delta(\pi)}}.$$

Consequence

If $\Delta(\pi) > 0$ then 100% of conics in the family are pointless!

Enter Loughran–Smeets

In 2016, Loughran–Smeets extended the definition of $\Delta(\pi)$ to fibrations $X \xrightarrow{\pi} \mathbb{P}^n$ where X is smooth and π has geometrically integral generic fibre.

In general, the fibres need not satisfy the Hasse principle.

Given the difficulty of understanding the obstruction in general, we instead ask when is the fibre *everywhere locally soluble*?

Define

$$N_{\text{loc}}(\pi, B) = \#\{x \in \mathbb{P}^n(\mathbb{Q}) : H(x) \leqslant B \text{ and } \pi^{-1}(x)(\mathbb{Q}_v) \neq \varnothing \, \forall v\}.$$

Then Loughran–Smeets establish

$$N_{\text{loc}}(\pi, B) \ll \frac{B}{(\log B)^{\Delta(\pi)}}.$$

Q.

Is this upper bound sharp?

The LRS constant

Inspired by the Manin–Peyre conjecture, Loughran–R.–Sofos predict an asymptotic formula for this counting function.

Conjecture (Special case of $\mathbb{P}^n_{\mathbb{Q}}$ base)

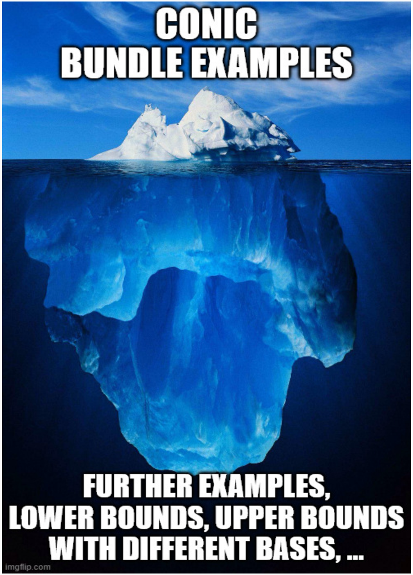
Let $X/\mathbb{Q}$ be smooth projective variety and $X \xrightarrow{\pi} \mathbb{P}^n_{\mathbb{Q}}$ a dominant morphism with geom. integral generic fibre such that

- *π admits a smooth fibre that is ELS,*
- *Every fibre over a codim 1 point has an irred. component of multiplicity 1,*

Then, we have $N_{loc}(\pi, B) \sim c_\pi \frac{B}{(\log B)^{\Delta(\pi)}}$ as $B \to \infty$.

The conjecture is stated for more general fibrations $X \xrightarrow{\pi} Y$ over number fields, but extra conditions are required.

Some Examples



Serre's conics

- $\{X^2 + Y^2 = tZ^2\} \to \mathbb{P}^1$.
  How frequently is a rational number a sum of two squares?
    - Asymptotic: ≈ Landau 1908, Loughran–R.–Sofos 2022(/2026+).
- $\{aX^2 + bY^2 + cZ^2 = 0\} \to \mathbb{P}^2$.
  Planar diagonal conics.
    - Lower bound: Guo 1995, Hooley 1993.
    - Asymptotic: Loughran–R.–Sofos 2022.
- $\{aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0\} \to \mathbb{P}^5$.
  Universal family of planar conics.
    - Lower bounds: Hooley 2007.
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[1] arXiv:2603.05849

Honourable mentions

- For $d \geqslant 3$, Koymans–Paterson–Santens–Shute (2025) settle the conjecture for the family

  $$\{aX^d + bY^d + cZ^d = 0\} \xrightarrow{\pi} \mathbb{P}^2$$

  using related techniques. Note in this case we can only discuss local solubility.
- Wilson (2024) provides an asymptotic for solubility of fibres in the family

  $$\{aX^2 + bY^2 + cZ^2 + dW^2 = 0\} \xrightarrow{\pi} \{ab = cd\}.$$

  This doesn’t fall under the remit of the LRS conjecture and demonstrates that loosening the geometric conditions therein leads to weird behaviour! It also demonstrates how thin set phenomena can arise in this problem.

More families of conics

- Loughran/Loughran–Takloo-Bighash–Tanimoto (2018/2020): Products of certain conics over toric varieties under anisotropic tori/wonderful compactifications of adjoint semisimple algebraic groups.
- Sofos–Visse-Martindale (2021): $\{X^2 + Y^2 = f_1(\mathbf{t})Z^2\} \xrightarrow{\pi} \{f_2(\mathbf{t}) = 0\} \subset \mathbb{P}^n$ where $\deg(f_1) = \deg(f_2)$ and $n > 3(d-1)2^d$.
- Da Silva (2025): $\{f_1(\mathbf{t})X^2 + f_2(\mathbf{t})Y^2 = f_3(\mathbf{t})Z^2\} \xrightarrow{\pi} \mathbb{P}^n$ where $\deg(f_i) = d$ and $n > 5(d-1)2^{d+1}$.
- Destagnol–Lyczak–Sofos (2025): $\{X^2 + DY^2 = f_1(\mathbf{t}) \cdots f_R(\mathbf{t})Z^2\} \xrightarrow{\pi} \mathbb{P}^n$ where $\deg(f_i) = d$ and $n > R(R+1)2^{d-1}(d-1)$.
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Many conics

In Chan–Koymans–R., we study systems of the form

$$\begin{array}{ccccccc} M_{1,1}(t_1,\ldots,t_n)X_1^2 & + & M_{1,2}(t_1,\ldots,t_n)Y_1^2 & + & M_{1,3}(t_1,\ldots,t_n)Z_1^2 & = & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ M_{m,1}(t_1,\ldots,t_n)X_m^2 & + & M_{m,2}(t_1,\ldots,t_n)Y_m^2 & + & M_{m,3}(t_1,\ldots,t_n)Z_m^2 & = & 0, \end{array}$$

where $M_{i,j}$ are squarefree monomials in $(t_1,\ldots,t_n)$.

- e.g. $m = 1, n = 2$ and $M_{1,i} = t_i$;

$$t_1X^2 + t_2Y^2 + t_3Z^2 = 0.$$

- e.g.

$$\begin{aligned} t_1t_2X_1^2 + t_3t_4Y_1^2 &= Z_1^2 \\ t_1t_3X_2^2 + t_2t_4Y_2^2 &= Z_2^2. \end{aligned}$$

The subordinate Brauer group and counting

Given a field k with char. $\neq 2$, there is a 1:1 correspondence

$$\{\text{Conics } aX^2+bY^2 = Z^2/k\}/\sim \longleftrightarrow \{\text{Quaternion algebras}(a,b)_k\}/\sim$$

The conic has a k point $\iff$ the quaternion algebra is split.
Given a quaternion algebra $\alpha = (a,b)_{\mathbb{Q}(t_0,\dots,t_n)} \in \mathrm{Br}\mathbb{Q}(\mathbb{P}^n)$ and $x \in \mathbb{P}^n(\mathbb{Q})$ not in the ramification locus of α, then there is a specialisation $\alpha_x = (a(x), b(x))_{\mathbb{Q}} \in \mathrm{Br}\mathbb{Q}$.

Counting how often $a(x)X^2 + b(x)Y^2 = Z^2$ has a $\mathbb{Q}$-point is equivalent to counting how often the specialisation α_x splits in $\mathrm{Br}\mathbb{Q}$.

(J.-P. Serre - *Spécialisation des éléments de* $\mathrm{Br}_2(\mathbb{Q}(T_1,\dots,T_n))$.)

Subordinate classes in a conic bundle

For smooth varieties V, Grothendieck's purity theorem gives

$$0 \to \operatorname{Br} V \to \operatorname{Br} \mathbb{Q}(V) \xrightarrow{\oplus \partial_D} \bigoplus_{D \in V^{(1)}} H^1_{\text{ét}}(\mathbb{Q}(D), \mathbb{Q}/\mathbb{Z}).$$

For $U \subset \mathbb{P}^n_{\mathbb{Q}}$ an open subset, let $\mathscr{B} = \{\beta_1, \ldots, \beta_n\} \subset \operatorname{Br}(U)$.

Definition

A class $b \in \operatorname{Br} \mathbb{Q}(\mathbb{P}^n)$ is *subordinate* to $\mathscr{B}$ if for each codimension one point D the residue $\partial_D(b)$ is contained within the set $\partial_D(\langle \mathscr{B} \rangle)$ of residues of classes in $\operatorname{Br} U$ generated by the classes of $\mathscr{B}$.

If $X = C_1 \times \cdots \times C_m \xrightarrow{\pi} \mathbb{P}^n$ is a product of conic bundles then let $\mathscr{B} = \{\alpha_1, \ldots, \alpha_m\}$ where α_i is the quaternion algebra associated to C_i. Then we write $\operatorname{Br}_{\text{Sub}}(\pi)$ for the set of $b \in \operatorname{Br} \mathbb{Q}(\mathbb{P}^n)$ subordinate to $\mathscr{B}$.

e.g.

- For the motivating example, $\{aX^2 + bY^2 + cZ^2 = 0\} \xrightarrow{\pi} \mathbb{P}^2$ We have $\mathscr{B} = \{(-ac, -bc)_{\mathbb{Q}(a,b,c)}\}$ and $\mathrm{Br}_{\mathsf{Sub}}(\pi)/\mathrm{Br}\mathbb{Q} \cong \mathbb{Z}/2\mathbb{Z} = \langle (-ac, -bc)_{\mathbb{Q}(a,b,c)} \rangle$.

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  $\mathrm{Br}_{\mathrm{Sub}}(\pi)/\mathrm{Br}\mathbb{Q} \cong \mathbb{Z}/2\mathbb{Z} = \langle(-ac, -bc)_{\mathbb{Q}(a,b,c)}\rangle$.
- Consider the following product of conics

$$\begin{cases} t_1 t_2 X_1^2 + t_3 t_4 Y_1^2 = Z_1^2 \\ t_1 t_3 X_2^2 + t_2 t_4 Y_2^2 = Z_2^2 \end{cases} \xrightarrow{\pi} \mathbb{P}^3.$$

  Then $\mathscr{B} = \{(t_1 t_2, t_3 t_4)_{\mathbb{Q}(\mathbf{t})}, (t_1 t_3, t_2 t_4)_{\mathbb{Q}(\mathbf{t})}\}$.
  The class $(t_1 t_4, t_2 t_3)_{\mathbb{Q}(\mathbf{t})}$ is subordinate to $\mathscr{B}$.

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Question

The subordinate Brauer group appears in the final constant.
What role does it play in the counting?

The Friedlander–Iwaniec method

Suppose we want to count how frequently the equation $aX^2 + bY^2 = Z^2$ has a point as a, b vary through square-free coprime integers in a box of sidelength B.

By Hasse–Minkowski and Hilbert reciprocity, it suffices to understand solubility in $\mathbb{Q}_p$ for all primes p.

At primes not dividing ab, solubility comes for free.
If $p \mid a$ then $aX^2 + bY^2 = Z^2$ is soluble exactly when $\left(\frac{b}{p}\right) = +1$ and similarly for $p \mid b$.

The Friedlander–Iwaniec method

Therefore the count can be computed via the following sum

$$\sum_{\substack{a,b\in\mathbb{Z}\text{ sq.free.}\\ \gcd(a,b)=1\\ |a|,|b|\leqslant B}} \mathbf{1}_{C_{a,b}(\mathbb{Q})\neq\varnothing} = \sum_{\substack{a,b\in\mathbb{Z}\text{ sq.free.}\\ \gcd(a,b)=1\\ |a|,|b|\leqslant B}} \prod_{p\mid ab} \frac{1}{2}\left(1+\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{b}{p}\right)\right)$$

$$= \sum_{|a|,|b|\leqslant B} \frac{\mu^2(ab)}{2^{\omega(ab)}} \prod_{p\mid ab}\left(1+\left(\frac{a}{p}\right)\right)\left(1+\left(\frac{b}{p}\right)\right)$$

$$= \sum_{|a|,|b|\leqslant B} \frac{\mu^2(ab)}{2^{\omega(ab)}} \sum_{\substack{d_1\mid a\\ d_2\mid b}} \left(\frac{a}{d_2}\right)\left(\frac{b}{d_1}\right)$$

$$= \sum_{\substack{|d_1 d_1'|\leqslant B\\ |d_2 d_2'|\leqslant B}} \frac{\mu^2(d_1 d_1' d_2 d_2')}{2^{\omega(d_1 d_1' d_2 d_2')}} \left(\frac{d_1 d_1'}{d_2}\right)\left(\frac{d_2 d_2'}{d_1}\right).$$

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$$\sum_{\substack{|d_1 d_1'| \leqslant B \\ |d_2 d_2'| \leqslant B}} \frac{\mu^2(d_1 d_1' d_2 d_2')}{2^{\omega(d_1 d_1' d_2 d_2')}} \left(\frac{d_1 d_1'}{d_2}\right) \left(\frac{d_2 d_2'}{d_1}\right)$$

The Legendre symbol $\left(\frac{c}{d}\right)$ is a Dirichlet character.

By fixing two of the variables, the resulting summand is a bilinear sum of characters. When the character is non-principal, we get cancellation in this sum $\rightsquigarrow$ Error term.

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However, if $d_1 = d_2 = 1$ then there is no character $\rightsquigarrow \sum_{|d_1'|,|d_2'| \leqslant B} \frac{\mu^2(d_1' d_2')}{2^{\omega(d_1' d_2')}} \asymp \frac{B^2}{\log B}$.

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Similarly, if $d_1' = d_2' = 1$ then

$\rightsquigarrow \sum_{|d_1'|, |d_2'| \leqslant B} \frac{\mu^2(d_1' d_2')}{2^{\omega(d_1' d_2')}} \left(\frac{d_1}{d_2}\right) \left(\frac{d_2}{d_1}\right) \asymp \frac{B^2}{\log B}$.

By quadratic reciprocity, this combination of characters doesn't oscillate! We say that 1 and 2 are *linked indices*.

Linked indices

In general, we have a more complicated version of the same picture.

- Detect local solubility using Legendre symbols.
- Expand the product over local indicator functions into a sum over divisors.
- Most of the time this sum exhibits cancellation.
- Main terms arise when there is a conspiracy between the variables that kills all the characters.

In Chan–Koymans–R., we develop complicated combinatorial machinery to understand when such conspiracies arise. This data is purely combinatorial and encodes when there is cancellation among complicated products of Legendre symbols.

A surprising connection

Given a finite set S, we can view the power set 2^S as an $\mathbb{F}_2$ vector space of dimension $\#S$. In order to understand when main terms arise, we construct “allowable” subsets $V_S \subset 2^{[n]\cup\{-\}}$ for every $\varnothing \subset S \subset [n]$. A main term will correspond to a map $f : 2^{[n]} \to 2^{[n]\cup\{-\}}$ satisfying:

- $f(\{i\}) \in V_{\{i\}}$ for all $i \in [n]$,
- f is linear,
- f is alternating, i.e. $|\{i\} \cap f(\{j\})| = |\{j\} \cap f(\{i\})|$ & $|\{i\} \cap f(\{i\})| = 0$.

It turns out that this data is in 1:1 correspondence with elements of $\mathrm{Br}_{\mathsf{Sub}}(\pi)$.

$$f \mapsto \sum_{i,j\in[n]} |\{i\} \cap f(\{j\})|(t_i, t_j)$$

e.g.

$$\begin{cases} t_1 t_2 X_1^2 + t_3 t_4 Y_1^2 = Z_1^2 \\ t_1 t_3 X_2^2 + t_2 t_4 Y_2^2 = Z_2^2 \end{cases}$$

$\mathscr{B} = \{(t_1 t_2, t_3, t_4)_{\mathbb{Q}(\mathbf{t})}, (t_1 t_3, t_2 t_4)_{\mathbb{Q}(\mathbf{t})}\}$.

When performing the counting, one encounters (among others) a main term of the form

$$\sum_{|d_i| \leqslant B} \frac{\mu^2(d_1 d_2 d_3 d_4)}{2^{\omega(d_1 d_2 d_3 d_4)}} \left(\frac{d_2 d_3}{d_1 d_4}\right) \left(\frac{d_1 d_4}{d_2 d_3}\right).$$

These indices are linked! The associated function is defined by

$$f(\{1\}) = f(\{4\}) = \{2, 3\}$$
$$f(\{2\}) = f(\{3\}) = \{1, 4\}.$$

This corresponds to the subordinate Brauer class $(t_1 t_4, t_2 t_3)_{\mathbb{Q}(\mathbf{t})}$.

Summary

- When counting how often conics are soluble, we expect an asymptotic similar to the Manin–Peyre conjecture.
- This asymptotic is now known for all the examples in Serre's original paper.
- Euler products arise with distinct local factors at a positive proportion of primes.
- A robust method for producing these asymptotics exists via character sum analysis.
- The leading constant will be a sum of terms corresponding to linked indices.
- Surprisingly, these are parametrised by subordinate Brauer classes.