RATIONAL POINTS ON KUMMER VARIETIES AND SWINNERTON-DYER'S METHOD

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Abstract. These lectures are designed to serve as an introduction to a method of Swinnerton-Dyer for studying the Hasse principle and Brauer—Manin obstruction for classes of varieties where the existence of rational points can be related to the existence of rational points on families of torsors under abelian varieties. Following Swinnerton-Dyer's original work, the method has been developed by Colliot-Thélène, Skorobogatov, Harpaz, Wittenberg and others, and examples of varieties it has been successfully applied to include certain K3 surfaces, cubic surfaces and quartic del Pezzo surfaces. We will primarily illustrate the method in the special case of Kummer surfaces. A sizeable portion of the course will be dedicated to the study of 2- and 4-Selmer groups in quadratic twist families, which forms a large part of the technical input to some of the results alluded to above.

Contents

1. Introduction

The aim of these notes is to give an introduction to Swinnerton-Dyer's method in the worked example of rational points on Kummer varieties, with a particular focus on (generalized) Kummer surfaces associated to Jacobians of genus 2 curves. A secondary aim is to discuss the behaviour of 2- and 4-Selmer groups in quadratic twist families.

While results on the existence of rational points established via Swinnerton-Dyer's method are typically conditional on finiteness of auxiliary Tate–Shafarevich groups and, sometimes, Schinzel's hypothesis, the method remains important since there are many varieties (in particular classes of K3 surfaces) where no other results or methods are known. In particular, the method can give evidence towards conjectures on the sufficiency of the Brauer–Manin obstruction for certain classes of varieties (for which see [CT03], [Sko09, p. 77] and references therein).

Instead of attempting to start with a general introduction to the method, we focus on some relevant theory and techniques before specialising to the case of Kummer surfaces. Works carrying out and developing Swinnerton-Dyer's method include [SD95, CTSSD98, SSD05, SD00, SD01, CT01, Wit07, HS16, Har19, MS24, Mor25]. We point to the introductions of [CT01] and [Har19] for an overview of the method, the context in which it sits, the role of many of the cited works, and additional references.

Date: 29th April 2026.

1.1. **Acknowledgements.** These are notes from a mini-course given as part of the Thematic Programme on Rational Points, Algebraic Cycles and the Local-Global Principle at the Lodha Mathematical Sciences Institute, Mumbai, April 2026. I would like to thank the organisers of that programme, Jean-Louis Colliot-Thélène, Parimala Raman and Federico Scavia, for the opportunity to give the course. I would also like to thank the Lodha institute for the excellent hospitality and working conditions provided, and everyone who attended the course for their questions and comments throughout. Finally, I would like to thank Jean-Louis Colliot-Thélène and Federico Scavia for helpful comments on an earlier version of these notes, and Jef Laga for discussions related to Theorem 2.6.

1.2. **Notation/background.** For a field k, we will denote by $\bar{k}$ a fixed algebraic closure of k, $k^s \subseteq \bar{k}$ the separable closure of k in $\bar{k}$, and $G_k = \mathrm{Gal}(k^s/k)$ the absolute Galois group of k.

For a smooth projective variety X over k we write $\mathrm{Br}(X)$ for the Brauer group of X and set

$$\mathrm{Br}_1(X) = \ker\big(\mathrm{Br}(X) \to \mathrm{Br}(X_{k^s})\big).$$

We also write $\mathrm{Br}_0(X)$ for the image of $\mathrm{Br}(k)$ in $\mathrm{Br}(X)$. If $H^3(k, k^{s,\times}) = 0$ (e.g. if k is a number field) then we have an isomorphism $\mathrm{Br}_1(X)/\mathrm{Br}_0(X) \cong H^1\big(k, \mathrm{Pic}(X_{\bar{k}})\big)$; see [CTS21, Proposition 4.3.2] for details. In the case that k is a number field, for a place v of k we denote by k_v the completion of k at v. We say that X is *everywhere locally soluble* if $X(k_v) \neq \emptyset$ for all places v of k. Equivalently, X is everywhere locally soluble if $X(\mathbb{A}_k) \neq \emptyset$, where $\mathbb{A}_k$ denotes the Adele ring of k. We say that X satisfies the *Hasse principle* if X being everywhere locally soluble implies that $X(k) \neq \emptyset$.

We now briefly define the Brauer–Manin obstruction to the Hasse principle, referring to [CTS21, Chapter 12] for details and examples. For each place v of k, we denote by $\mathrm{Br}(k_v)$ the Brauer group of k_v and by $\mathrm{inv}_v : \mathrm{Br}(k_v) \hookrightarrow \mathbb{Q}/\mathbb{Z}$ the local invariant map. There is a natural pairing $X(\mathbb{A}_k) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z}$ given by

$$\big((x_v)_v, \alpha\big) \mapsto \sum_{v \text{ place of } k} \mathrm{inv}_v \alpha(x_v),$$

where $\alpha(x_v) \in \mathrm{Br}(k_v)$ is the result of evaluating $\alpha \in \mathrm{Br}(X)$ at the local point $x_v \in X(k_v)$. We denote by $X(\mathbb{A}_k)^{\mathrm{Br}(X)}$ the subset of $X(\mathbb{A}_k)$ consisting of elements pairing trivially with $\mathrm{Br}(X)$. By reciprocity for the Brauer group of k, we have $X(k) \subseteq X(\mathbb{A}_k)^{\mathrm{Br}(X)}$. In general, one can have $X(\mathbb{A}_k)^{\mathrm{Br}(X)} = \emptyset$ even if $X(\mathbb{A}_k) \neq \emptyset$. In this case, one says that there is a *Brauer–Manin obstruction* to the Hasse principle on X. We say that the Brauer–Manin obstruction is the only obstruction to the Hasse principle on X if $X(\mathbb{A}_k)^{\mathrm{Br}(X)} \neq \emptyset$ implies that $X(k) \neq \emptyset$.

2. The Cassels–Tate pairing

Let k be a number field and A/k an abelian variety. The Galois cohomology group $H^1(k, A)$ classifies isomorphism classes of A-torsors X/k. The Tate–Shafarevich group of A is defined as

$$Ш(A) = \ker\left(H^1(k, A) \to \prod_{v \text{ place of } k} H^1(k_v, A)\right).$$

It is the subgroup of $H^1(k, A)$ consisting of elements represented by everywhere locally soluble torsors, i.e. torsors X under A such that $X(\mathbb{A}_k) \neq \emptyset$. A major open conjecture is that $Ш(A)$ is finite.

The Cassels–Tate pairing is a bilinear pairing

$$\langle\ ,\ \rangle_{\mathrm{CT}} : Ш(A) \times Ш(A^\vee) \to \mathbb{Q}/\mathbb{Z},$$

where $A^\vee$ denotes the dual abelian variety. It has several equivalent definitions; see [PS99]. One definition, the *homogeneous space definition*, is as follows.

Definition 2.1 (Cassels–Tate pairing)**.** Let X be an everywhere locally soluble A-torsor, so that X represents a class in $Ш(A)$. We have a canonical isomorphism of G_k-modules $\mathrm{Pic}^0(X_{\bar{k}}) \cong \mathrm{Pic}^0(A_{\bar{k}}) = A^\vee(\bar{k})$. This induces a homomorphism

$$\Phi_X : Ш(A^\vee) \to H^1\big(k, \mathrm{Pic}(X_{\bar{k}})\big) \cong \mathrm{Br}_1(X)/\mathrm{Br}_0(X).$$

We note that since elements of $Ш(A^\vee)$ are everywhere locally trivial, the image of this map consists of elements of $\mathrm{Br}_1(X)$ that everywhere locally lie in the image of $\mathrm{Br}(k_v)$ in $\mathrm{Br}(X_{k_v})$. Now take $\eta \in Ш(A^\vee)$ and let $\alpha \in \mathrm{Br}_1(X)$ be any element representing the class of $\Phi_X(\eta)$. Define

$$\langle [X], \eta \rangle_{\mathrm{CT}} = \sum_{v \text{ place of } k} \mathrm{inv}_v \alpha(x_v), \tag{2.2}$$

where $x_v \in X(k_v)$ is any choice of k_v-point on X.

Remark 2.3. The right hand side of (2.2) is independent of the choice of x_v since α is everywhere locally in the image of $\mathrm{Br}(k_v)$, hence the evaluation map $X(k_v) \to \mathrm{Br}(k_v)$ given by $x_v \mapsto \alpha(x_v)$ is constant. Further, by reciprocity for the Brauer group of k, it is independent of the choice of representative $\alpha \in \mathrm{Br}_1(X)$ for $\Phi_X(\eta) \in \mathrm{Br}_1(X)/\mathrm{Br}_0(X)$.

The key property that the Cassels–Tate pairing has is that its left (resp. right) kernel is the maximal divisible subgroup of $Ш(A)$ (resp. of $Ш(A^\vee)$); see e.g. [Mil06, Chapter I.6]. If $Ш(A)$ is finite as expected then so is $Ш(A^\vee)$ and, in particular, the Cassels–Tate pairing is non-degenerate.

In his 1970 ICM talk Manin [Man70] defined what is now known as the Brauer–Manin obstruction, and observed the following.

Theorem 2.4 (Manin)**.** *Suppose that $Ш(A)$ is finite. Then the Brauer–Manin obstruction is the only obstruction to the Hasse principle for torsors under A.*

Proof. Let X be a torsor under A. We can assume that X is everywhere locally soluble, in which case X defines a class $[X] \in Ш(A)$. Suppose $X(k) = \emptyset$, so that $[X] \neq 0$. Since $Ш(A)$ is finite, the Cassels–Tate pairing is non-degenerate, hence there is $\eta \in Ш(A^\vee)$ with $\langle [X], \eta \rangle_{\mathrm{CT}} \neq 0$. Any lift $\alpha \in \mathrm{Br}_1(X)$ of $\Phi_X(\eta)$ gives a Brauer–Manin obstruction to the Hasse principle on X. $\square$

The basic aim of Swinnerton-Dyer's method is to (provide a framework allowing one to) 'propogate' this result to other classes of varieties. Before saying any more about this, we discuss another relevant part of the theory of the Cassels–Tate pairing.

2.1. **The obstruction to being alternating.** Suppose now that A admits a principal polarisation $\lambda : A \to A^\vee$.[1] Via λ we identify A with $A^\vee$ and thus view the Cassels–Tate pairing as a $\mathbb{Q}/\mathbb{Z}$-valued bilinear pairing on $Ш(A)$. As shown by Flach [Fla90], this pairing is antisymmetric. That is, for all $x, y \in Ш(A)$, we have $\langle x, y \rangle_{\mathrm{CT}} = -\langle y, x \rangle_{\mathrm{CT}}$. In particular, we have $2\langle x, x \rangle_{\mathrm{CT}} = 0$ for all $x \in Ш(A)$. One can ask whether, moreover, the Cassels–Tate pairing is alternating (i.e. $\langle x, x \rangle_{\mathrm{CT}} = 0$ for all $x \in Ш(A)$). This question was given a beautiful answer by Poonen and Stoll in [PS99]. We recall their result.

We have a short exact sequence of G_k-modules

$$0 \to A^\vee(\bar{k}) \to \mathrm{Pic}(A_{\bar{k}}) \to NS(A_{\bar{k}}) \to 0. \tag{2.5}$$

The associated long exact sequence for Galois cohomology induces a connecting map

$$\delta : NS(A_{\bar{k}})^{G_k} \to H^1(k, A^\vee).$$

The principal polarisation λ gives a class in $NS(A_{\bar{k}})^{G_k}$, so we obtain a class $\delta(\lambda) \in H^1(k, A^\vee)$. Denote by c the image of this class under the isomorphism $H^1(k, A^\vee) \cong H^1(k, A)$ induced by λ.

Theorem 2.6 (Poonen–Stoll, [PS99] Theorem 5)**.** *The element c lies in $Ш(A)[2]$. Moreover, for all $x \in Ш(A)$, we have $\langle x, x \rangle_{\mathrm{CT}} = \langle x, c \rangle_{\mathrm{CT}}$. In particular, $\langle\ ,\ \rangle_{\mathrm{CT}}$ is alternating if and only if c lies in the maximal divisible subgroup of $Ш(A)$.*

[1] For each $\mathcal{L} \in \mathrm{Pic}(A_{\bar{k}})$, one has an associated homomorphism $\phi_{\mathcal{L}} : A_{\bar{k}} \to A^\vee_{\bar{k}}$ given by $\phi_{\mathcal{L}}(x) = \tau_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$, where τ_x is translation by $x \in A(\bar{k})$; the homomorphism depends only on the class of $\mathcal{L}$ in the Néron–Severi group $NS(A_{\bar{k}})$. A homomorphism $\lambda : A \to A^\vee$ is called a *polarisation* if its base-change to $\bar{k}$ is equal to $\phi_{\mathcal{L}}$ for some ample $\mathcal{L} \in \mathrm{Pic}(A_{\bar{k}})$. A polarisation is *principal* if it is an isomorphism.

Sketch of proof. We do not comment on the claim that c lies in $Ш(A)[2]$, save to remark that this uses the local duality theory of abelian varieties (that c is 2-torsion follows from the discussion preceding Proposition 2.11 below). However, we indicate why the identity $\langle x,x\rangle_{\mathrm{CT}}=\langle x,c\rangle_{\mathrm{CT}}$ holds.

Let $x\in Ш(A)$ be represented by an everywhere locally soluble torsor X under A. As is explained nicely in [Lag25, Section 2.4] (see also the references therein), as well as the canonical isomorphism of G_k-modules $\mathrm{Pic}^0(X_{\bar{k}})\cong A^\vee(\bar{k})$ noted above, we also have an isomorphism of G_k-modules $NS(X_{\bar{k}})\cong NS(A_{\bar{k}})$ (since translations act trivially on the Néron–Severi group of $A_{\bar{k}}$), hence a short exact sequence of G_k-modules

$$0\to A^\vee(\bar{k})\to \mathrm{Pic}(X_{\bar{k}})\to NS(A_{\bar{k}})\to 0. \tag{2.7}$$

We caution that, in general, we *need not* have an isomorphism of G_k-modules $\mathrm{Pic}(X_{\bar{k}})\cong\mathrm{Pic}(A_{\bar{k}})$, so this sequence can be genuinely different from the sequence (2.5) used above to define the class c. Denoting by $\delta_X: NS(A_{\bar{k}})^{G_k}\to H^1(k,A^\vee)$ the connecting map arising from (2.7), we obtain a class $\delta_X(\lambda)\in H^1(k,A^\vee)$ whose image in $H^1(k,\mathrm{Pic}(X_{\bar{k}}))$ is trivial. By studying the difference between the Galois action on $\mathrm{Pic}(X_{\bar{k}})$ and the Galois action on $\mathrm{Pic}(A_{\bar{k}})$ (which can be understood in terms of a 1-cocycle representing x by fixing a point $z\in X(\bar{k})$ and using this to define an isomorphism $\phi: A_{\bar{k}}\to X_{\bar{k}}$ given by $a\mapsto a+z$, hence an isomorphism $(\phi^{-1})^*:\mathrm{Pic}(A_{\bar{k}})\to\mathrm{Pic}(X_{\bar{k}})$; cf. also [Ale02, Theorem 3.0.3(i)]), one computes that

$$\delta_X(\lambda)=\delta(\lambda)-\lambda(x)=\lambda(c-x),$$

where δ is the connecting map associated to (2.5). Thus $\lambda(c-x)$ maps to zero in $H^1(k,\mathrm{Pic}(X_{\bar{k}}))$. The definition of the Cassels-Tate pairing then shows that we have

$$\langle x,c-x\rangle_{\mathrm{CT}}=0,$$

which rearranges to the sought identity. □

Corollary 2.8 ([PS99] Theorem 8 and Corollary 9)**.** *Suppose that $Ш(A)$ is finite. If $\langle c,c\rangle_{\mathrm{CT}}=0$ then $Ш(A)\cong T\times T$ for some finite abelian group T. In particular, $\dim_{\mathbb{F}_2}Ш(A)[2]$ is even and $Ш(A)$ has square order. Conversely, if $\langle c,c\rangle_{\mathrm{CT}}=1/2$, then $Ш(A)\cong\mathbb{Z}/2\mathbb{Z}\times T\times T$ for some finite abelian group T, $\dim_{\mathbb{F}_2}Ш(A)[2]$ is odd and the order of $Ш(A)$ is twice a square.*

Proof. In general, suppose we have a finite abelian group M equipped with a non-degenerate antisymmetric bilinear pairing $P: M\times M\to\mathbb{Q}/\mathbb{Z}$. The antisymmetry means that the map $x\mapsto P(x,x)$ is a homomorphism $M\to\frac{1}{2}\mathbb{Z}/\mathbb{Z}$, hence has the form $P(-,c)$ for a unique $c\in M[2]$. The map $\phi: M\to M$ given by

$$m\mapsto\begin{cases} m & P(m,c)=0,\\ m+c & P(m,c)=1/2,\end{cases}$$

is then checked to be a homomorphism, and the bilinear pairing $\widetilde{P}: M\times M\to\mathbb{Q}/\mathbb{Z}$ given by $\widetilde{P}(x,y)=P(x,\phi(y))$ checked to be alternating. Its kernel is $\ker(\phi)$, which is 0 if $P(c,c)=0$, and is $\langle c\rangle\cong\mathbb{Z}/2\mathbb{Z}$ if $P(c,c)=1/2$. In this latter case, we have $M\cong\langle c\rangle\oplus\langle c\rangle^\perp$, where the orthogonal complement is taken with respect to P and, moreover, $\widetilde{P}$ is a non-degenerate alternating pairing on $\langle c\rangle^\perp$. To conclude, recall that any finite abelian group admitting a non-degenerate alternating pairing is isomorphic to $T\times T$ for some finite abelian group T. We deduce that there is a finite abelian group T such that:

$$M\cong\begin{cases} T\times T & P(c,c)=0,\\ \mathbb{Z}/2\mathbb{Z}\times T\times T & P(c,c)=1/2.\end{cases}$$

Applying this with $M=Ш(A)$ and P the Cassels–Tate pairing gives the result. □

Remark 2.9. We stress that the condition $\langle c,c\rangle_{\mathrm{CT}}=0$ does not itself force the Cassels–Tate pairing to be alternating. Rather, as the proof of Corollary 2.8 shows, it guarantees the existence of *some* alternating pairing on $Ш(A)$ which is non-degenerate provided $Ш(A)$ is finite.

Remark 2.10. Let $Ш_{\mathrm{div}}(A)$ denote the maximal divisible subgroup of $Ш(A)$. Thus, the Cassels–Tate pairing gives a non-degenerate pairing on $Ш_{\mathrm{nd}}(A) := Ш(A)/Ш_{\mathrm{div}}(A)$. One still does not know unconditionally that $Ш_{\mathrm{nd}}(A)$ is finite. However, for all primes p, its p-part is known to be finite. Corollary 2.8 then holds unconditionally for $Ш_{\mathrm{nd}}(A)[2^\infty]$ (meanwhile, $Ш_{\mathrm{nd}}(A)[p^\infty]$ is the square of a finite abelian p-group for all odd primes p).

The class $\delta(\lambda) \in H^1(k, A^\vee)$ is the $A^\vee$-torsor associated to the G_k-set of line bundles $\mathcal{L} \in \mathrm{Pic}(A_{\bar{k}})$ inducing the given principal polarisation λ (that is, such that λ is given by $x \mapsto \tau_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$ where $\tau_x : A \to A$ is translation by x). This admits a natural refinement to an $A^\vee[2]$-torsor. Namely, a lift of $\delta(\lambda)$ to $H^1(k, A^\vee[2])$ is given by the class associated to the G_k-set of symmetric line bundles inducing λ. This gives a useful interpretation of the Poonen–Stoll class c as follows; see [PR11, Section 3.4] for details. Via the principal polarisation λ, we view the Weil pairing $e_2 : A[2] \times A^\vee[2] \to \mu_2$ as a pairing $A[2] \times A[2] \to \mu_2$, which by an abuse of notation we denote by e_2 also. It is non-degenerate, bilinear, alternating and G_k-equivariant. In particular (since $A[2]$ is 2-torsion) it is symmetric. A *quadratic refinement* of e_2 is a map $q : A[2] \to \mu_2$ such that

$$q(x+y)q(x)^{-1}q(y)^{-1} = e_2(x,y)$$

for all $x, y \in A[2]$. The G_k-set of quadratic refinements of the Weil pairing on $A[2]$ is a principal homogeneous space for $A[2]$, hence gives rise to a class $c_2 \in H^1(k, A[2])$. This class lifts $c \in H^1(k, A)$ to $H^1(k, A[2])$. Recall that, by definition, the preimage of $Ш(A)[2]$ in $H^1(k, A[2])$ is the 2-Selmer group $\mathrm{Sel}_2(A)$. We deduce the following result from the above discussion:

Proposition 2.11. *The class c_2 lies in $\mathrm{Sel}_2(A)$. Moreover, writing $G = \mathrm{Gal}(k(A[2])/k)$, the class c_2 arises via inflation from $H^1(G, A[2])$.*[2] *In particular, if G_k acts on $A[2]$ through the orthogonal group $O(q)$ of a quadratic refinement of the Weil pairing (e.g. if $A[2] \subseteq A(k)$) then $c = 0$. In this case, if $Ш(A)$ is finite then $\dim_{\mathbb{F}_2} Ш(A)[2]$ is even.*

2.2. **Quadratic twists.** Let k be a field, A/k a principally polarised abelian variety, and $\chi : G_k \to \mu_2$ a quadratic character. Denote by A^χ the quadratic twist of A by χ. That is, A^χ is the twist of A given by viewing χ as a 1-cocycle valued in $\{\pm 1\} \subseteq \mathrm{Aut}(A_{k^s})$. Thus, A^χ is equipped with a k^s-isomorphism $\phi : A \to A^\chi$ such that, for all $\sigma \in G_k$, $\phi^{-1} \circ \sigma\phi\sigma^{-1}$ is multiplication by $\chi(\sigma)$. In particular, A^χ is isomorphic to A over the quadratic extension cut out by χ.

It will be crucial in what follows to note that ϕ gives a Galois equivariant isomorphism $A[2] \cong A^\chi[2]$. Via this map, we will always identify $A^\chi[2]$ with $A[2]$. The principal polarisation on A descends to a principal polarisation on A^χ (since multiplication by -1 acts trivially on the Néron–Severi group of A) and the identification of $A[2]$ with $A^\chi[2]$ is compatible with the resulting Weil pairings.

Remark 2.12. Proposition 2.11 is particularly useful when considering the behaviour of Selmer/Tate–Shafarevich groups under quadratic twist. Indeed, the interpretation of c_2 via quadratic refinements of the Weil pairing shows that c_2 is invariant under quadratic twist. Thus, if G_k acts on $A[2]$ through the orthogonal group $O(q)$ of a quadratic refinement of the Weil pairing, and the Tate–Shafarevich groups of all quadratic twists of A are finite, then $\dim_{\mathbb{F}_2} Ш(A^\chi)[2]$ is even for every quadratic twist A^χ of A.

3. Remarks on Swinnerton-Dyer's method for Kummer varieties

The following is a situation in which one might hope that the existence of rational points on a given variety can be related to the existence of rational points on torsors under abelian varieties.

[2] Since the Galois action on $A[2]$ determines the Galois action on the set of quadratic refinements of e_2.

3.1. **Generalised Kummer varieties.** Let k be a field of characteristic different from 2 and let A/k be a principally polarised abelian variety of dimension at least 2. The Kummer variety X associated to A is a desingularisation of the quotient $X_{\mathrm{sing}} = A/\pm 1$. More specifically, X_{sing} is singular at the image of $A[2]$ and X is the blow up of X_{sing} at these points. When A has dimension 2, X is a $K3$ surface. Note that $X(k) \neq \emptyset$ since the same is true of A. However, there is a generalisation of this construction that yields varieties where the existence of k-points is much more interesting.

Fix a non-zero class $\alpha \in H^1(k, A[2])$ and denote by Y_α the corresponding 2-covering of A. Thus, Y_α is a torsor under A equipped with an involution ι compatible with multiplication by -1 on A. The generalised Kummer variety X_α associated to α is the blow up of the quotient $Y_\alpha/\langle\iota\rangle$ at the image of the fixed points of ι. It is a twist of X. Denote by $\widetilde{Y}_\alpha$ the blow up of Y_α at the fixed points of ι. Then ι extends uniquely to an involution $\widetilde{\iota}$ on $\widetilde{Y}_\alpha$ and $X_\alpha = \widetilde{Y}/\langle\widetilde{\iota}\rangle$.

Now let $\chi : G_k \to \mu_2$ be a quadratic character. We identify μ_2 with $\langle\iota\rangle \leq \mathrm{Aut}_{k^s}(Y_\alpha)$ and thus view χ as a 1-cocycle valued in $\mathrm{Aut}_{k^s}(Y_\alpha)$. Denote by Y_α^χ the corresponding quadratic twist of Y_α by χ. It is the 2-covering for A^χ arising from viewing α as an element of $H^1(k, A^\chi[2])$ via the identification $A[2] \cong A^\chi[2]$ discussed above. Since the construction of X_α involves taking the quotient by $\langle\iota\rangle$, the Kummer variety associated to Y_α^χ is (isomorphic over k to) X_α. Thus, for each quadratic character χ, we have a degree 2 morphism $\widetilde{Y}_\alpha^\chi \to X_\alpha$. Note, in particular, that if $Y_\alpha^\chi(k) \neq \emptyset$ for some χ then $X_\alpha(k) \neq \emptyset$. Conversely, if $X_\alpha(k) \neq \emptyset$, say $x \in X_\alpha(k)$, then the preimage of x under the quotient map $f : \widetilde{Y}_\alpha \to X_\alpha$ has order at most 2 and is stable under G_k. We deduce from this that, after a suitable quadratic twist, the fibre over x is defined over k. That is, there is a quadratic character χ such that $\widetilde{Y}_\alpha^\chi(k) \neq \emptyset$, which is the case if and only if $Y_\alpha^\chi(k) \neq \emptyset$. In this way, the existence of a k-point on X_α is equivalent to the existence of a k-point on Y_α^χ for some χ, reducing the study of k-points on X_α to the study of rational points in a family of torsors under abelian varieties.

3.2. **Basic strategy.** Now take k to be a number field. By a conjecture of Skorobogatov, one expects that the Brauer–Manin obstruction explains all failures of the Hasse principle for $K3$ surfaces, hence in particular Kummer surfaces. The Brauer group of generalised Kummer varieties is studied by Skorobogatov–Zarhin [SZ17]. Suppose we are in a situation where (we do not expect) the Brauer group to obstruct the Hasse principle on X_α. To avoid trivial cases, suppose also that $\alpha \neq 0$ and $X_\alpha(\mathbb{A}_k) \neq \emptyset$. From the discussion above this latter assumption implies, in particular, that for each place v of k there is a quadratic character $\chi_v : G_{k_v} \to \mu_2$ such that $Y_\alpha^{\chi_v}(k_v) \neq \emptyset$. The basic strategy (which often turns out to be too optimistic) for establishing the existence of a k-point on X_α, conditional on finiteness of Tate–Shafarevich groups, goes as follows:

(1) find a global quadratic character $\chi : G_k \to \mu_2$ such that $Y_\alpha^\chi(\mathbb{A}_k) \neq \emptyset$. Equivalently, such that $\alpha \in \mathrm{Sel}_2(A^\chi)$.

(2) construct a sequence $\chi = \chi_1, \chi_2, ..., \chi_n$ of quadratic characters such that passing from χ_i to χ_{i+1} induces a controlled change on the 2-Selmer group, say
$$\mathrm{Sel}_2(A^\chi) \supsetneq \mathrm{Sel}_2(A^{\chi_2}) \supsetneq \mathrm{Sel}_2(A^{\chi_3}) \supsetneq \cdots \supsetneq \mathrm{Sel}_2(A^{\chi_n}) = \langle\alpha\rangle\,.$$
The idea here is to alter χ in such a way that α remains in the 2-Selmer group at each step, but such that the 2-Selmer group of $A^{\chi_{i+1}}$ is strictly smaller than that of A^{χ_i}. If one can always do this, one eventually arrives at a situation where $\mathrm{Sel}_2(A^{\chi_n})$ is 1-dimensional, generated by α.[3]

(3) We have a surjective map $\mathrm{Sel}_2(A^{\chi_n}) \to Ш(A^{\chi_n})[2]$. If we are in a situation where finiteness of $Ш(A^{\chi_n})$ implies that $Ш(A^{\chi_n})[2]$ is even dimensional (e.g. if G_k acts on $A[2]$ through the orthogonal group of some quadratic refinement of the Weil pairing) then, conditional on finiteness of Tate–Shafarevich groups, we must have $Ш(A^{\chi_n})[2] = 0$. Thus, the class of α in $Ш(A^{\chi_n})$ is trivial, hence $Y^\alpha(k) \neq \emptyset$. Thus also $X_\alpha(k) \neq \emptyset$.

[3] Note that $\dim \mathrm{Sel}_2(A^\chi) \geq \dim A(k)[2]$ for all χ, so the presence of rational 2-torsion can stop this step being possible as written. However, one can consider variants where $\mathrm{Sel}_2(A^{\chi_n})$ is generated by α and the image of the k-rational 2-torsion points, etc.

We will see later some situations where the strategy works in exactly this way. However, there are many others where this strategy runs into difficulty and additional tricks must be found to save it. A key recent innovation, due to Harpaz [Har19], is to incorporate a second descent step (more about this later).

4. Selmer groups

We begin with the abstract framework of Selmer structures, before specialising to Selmer groups of abelian varieties. In this section, k denotes a number field.

4.1. **Selmer structures.** Let M be a finite G_k-module with exponent n. Define the *dual* of M, $M^\vee = \mathrm{Hom}(M, \mu_n)$. Define, for v a non-archimedean place of k, the *unramified subspace*

$$H^1_{\mathrm{ur}}(k_v, M) := \ker\left(H^1(k_v, M) \xrightarrow{\mathrm{res}} H^1(k_v^{\mathrm{ur}}, M)\right),$$

where k_v^{ur} denotes the maximal unramified extension of k_v. The inflation-restriction exact sequence means that each class in $H^1_{\mathrm{ur}}(k_v, M)$ can be represented by a 1-cocycle that factors through the natural surjection $G_{k_v} \to \mathrm{Gal}(k_v^{\mathrm{ur}}/k_v)$.

For each place v of k, define the *local Tate pairing*

$$\langle\ ,\ \rangle_v : H^1(k_v, M) \times H^1(k_v, M^\vee) \longrightarrow H^2(k_v, \mu_n) = \mathrm{Br}(K_v)[2] \hookrightarrow \mathbb{Q}/\mathbb{Z}$$

given by composition of cup-product and the local invariant map.

Theorem 4.1 (Tate local duality)**.** *For each place v of k, the local Tate pairing is non-degenerate. Moreover, for each nonarchimedean place $v \nmid n$ such that the inertia group acts trivially on M, $H^1_{\mathrm{ur}}(k_v, M)$ and $H^1_{\mathrm{ur}}(k_v, M^\vee)$ are orthogonal complements.*

Proof. See [NSW08, Corollary 7.2.6] for nonarchimedean v and op. cit. Theorem 7.2.17 for archimedean v. The claim about unramified subspaces is op. cit. Theorem 7.2.15. □

Definition 4.2 (Selmer structure)**.** A *Selmer structure* $\mathcal{L} = \{\mathcal{L}_v\}_v$ for M is a collection of subspaces $\mathcal{L}_v \subseteq H^1(k_v, M)$ for each place v of k, such that $\mathcal{L}_v = H^1_{\mathrm{ur}}(k_v, M)$ for all but finitely many v. The associated *Selmer group* $\mathrm{Sel}_{\mathcal{L}}(M)$ is defined as

$$\mathrm{Sel}_{\mathcal{L}}(M) = \{x \in H^1(k, M)\ :\ \mathrm{res}_v(x) \in \mathcal{L}_v \text{ for all places } v \text{ of } k\}.$$

The assumption that $\mathcal{L}_v = H^1_{\mathrm{ur}}(k_v, M)$ for all but finitely many places ensures that $\mathrm{Sel}_{\mathcal{L}}(M)$ is finite. Define the *dual Selmer structure* $\mathcal{L}^\perp = \{\mathcal{L}_v^\perp\}$ for $M^\vee$ by taking $\mathcal{L}_v^\perp$ to be the orthogonal complement of $\mathcal{L}_v$ for each place v.

4.2. **Selmer groups of abelian varieties.** Let A be a principally polarised abelian variety over k. We now specalise to the case that $M = A[n]$ for some integer $n \geq 2$. The assumption that A is principally polarised implies that $A[n]$ is self-dual via the Weil pairing. We thus view the local Tate pairing as a (non-degenerate, bilinear) self-pairing

$$\langle\ ,\ \rangle_v : H^1(k_v, A[n]) \times H^1(k_v, A[n]) \to \mathbb{Q}/\mathbb{Z}.$$

For each place v, taking Galois cohomology of the short exact sequence

$$0 \to A[n] \to A \xrightarrow{n} A \to 0 \tag{4.3}$$

induces a connecting map $\delta_v : A(k_v)/nA(k_v) \hookrightarrow H^1(k_v, A[n])$.

Lemma 4.4. *Let v be a place of k.*

(i) *$\delta_v(A(k_v))$ is its own orthogonal complement with respect to the local Tate pairing,*
(ii) *Suppose that $v \nmid n$ is a nonarchimedean place of k. Then*

$$\#\delta(A(k_v)) = \#A(k_v)[n].$$

Moreover, if A has good reduction at v, then $\delta(A(k_v)) = H^1_{\mathrm{ur}}(k_v, A[n])$.

As a result of Lemma 4.4, setting $\mathcal{L}_v = \delta(A(k_v))$ gives a self-dual Selmer structure for $A[n]$. The resulting Selmer group is the n-Selmer group of A, $\mathrm{Sel}_n(A) \subseteq H^1(k, A[n])$. Taking Galois cohomology of the sequence (4.3) over both k and k_v, we deduce the existence of the well-known short exact sequence

$$0 \to A(k)/nA(k) \to \mathrm{Sel}_n(A) \to \text{Ш}(A)[n] \to 0. \tag{4.5}$$

4.3. **2-Selmer groups in quadratic twist families.** Let $\chi : G_k \to \mu_2$ be a quadratic character. As above, we identify $A[2]$ with $A^\chi[2]$ as G_k-modules. In particular, we may (and always do!) view $\mathrm{Sel}_2(A^\chi)$ as a subgroup of $H^1(k, A[2])$. We can thus hope to compare $\mathrm{Sel}_2(A)$ and $\mathrm{Sel}_2(A^\chi)$ by comparing the relevant local conditions subgroups. Note that since the isomorphism $A[2] \cong A^\chi[2]$ is compatible with the Weil pairing, it identifies the local Tate pairings also.

Let v be a place of k. We will denote by δ_v the map $A(k_v)/2A(k_v) \to H^1(k_v, A[2])$ considered previously. For a quadratic character $\chi : G_k \to \{\pm 1\}$, we denote by δ_v^χ the composition

$$\delta_v^\chi : A^\chi(k_v)/2A^\chi(k_v) \to H^1(k_v, A^\chi[2]) \cong H^1(k_v, A[2])$$

given by composing the connecting map for A^χ with the isomorphism $A^\chi[2] \cong A[2]$ induced by ϕ^{-1}. We thus have two self-dual Selmer structures $\{\mathrm{im}(\delta_v)\}_v$ and $\{\mathrm{im}(\delta_v^\chi)\}_v$ on $A[2]$.

Lemma 4.6. *Let $v \nmid 2$ be a nonarchimedean place of good reduction for A. Let $\chi : G_k \to \mu_2$ be a quadratic character.*

(i) *If χ is unramified at v then*

$$\mathrm{im}(\delta_v) = H^1_{\mathrm{ur}}(k_v, A[2]) = \mathrm{im}(\delta_v^\chi),$$

(ii) *If χ is ramified at v then we have $\mathrm{im}(\delta_v^\chi) = \delta_v^\chi(A(k_v)[2])$. Moreover, we have*

$$\mathrm{im}(\delta_v) \cap \mathrm{im}(\delta_v^\chi) = 0.$$

Proof. (i). Since χ is unramified at v, both A and A^χ have good reduction at v and the result follows from Lemma 4.4 (ii).

(ii). Since A has good reduction at v, $A[4]$ is unramified. On the other hand, since χ is ramified at v, there is σ in the inertia group at v such that $\chi(\sigma) = -1$. Then σ acts a multiplication by -1 on $A^\chi[4]$. From this, we deduce that $H^0(k_v, A^\chi[4]) = A(k_v)[2]$. Thus, $\delta_v^\chi : A(k_v^{\mathrm{ur}})[2] \to H^1(k^{\mathrm{ur}}, A[2])$ is injective. In particular, $\delta_v^\chi : A(k_v)[2] \to H^1(k_v, A[2])$ is injective. Lemma 4.4 (ii) gives $\#\mathrm{im}(\delta_v^\chi) = \#A(k_v)[2]$, so we deduce that $\mathrm{im}(\delta_v^\chi) = \delta_v^\chi(A(k_v)[2])$. Finally, we have $\mathrm{im}(\delta_v) = H^1_{\mathrm{ur}}(k_v, A[2])$, so we must show that $H^1_{\mathrm{ur}}(k_v, A[2]) \cap \delta_v^\chi(A(k_v)[2]) = 0$. This follows from the injectivity of $\delta_v^\chi : A(k_v^{\mathrm{ur}})[2] \to H^1(k^{\mathrm{ur}}, A[2])$ noted previously. □

Let S be a finite set of places of k and let $\chi : G_k \to \mu_2$ be a quadratic character. Write

$$V_S = \mathrm{im}\left(\mathrm{Sel}_2(A) \overset{\oplus_{v\in S}\mathrm{res}_v}{\longrightarrow} \bigoplus_{v\in S} \frac{\mathrm{im}(\delta_v)}{\mathrm{im}(\delta_v) \cap \mathrm{im}(\delta_v^\chi)}\right),$$

and define V_S^χ similarly as

$$V_S^\chi = \mathrm{im}\left(\mathrm{Sel}_2(A^\chi) \overset{\oplus_{v\in S}\mathrm{res}_v}{\longrightarrow} \bigoplus_{v\in S} \frac{\mathrm{im}(\delta_v^\chi)}{\mathrm{im}(\delta_v) \cap \mathrm{im}(\delta_v^\chi)}\right).$$

The following result, stated and proved by Harpaz as [Har19, Lemma 3.27], has its origins in work of Mazur–Rubin [MR10, Section 3].

Proposition 4.7. *Suppose that $\mathrm{im}(\delta_v) = \mathrm{im}(\delta_v^\chi)$ for all $v \notin S$. Then*

$$\dim_{\mathbb{F}_2} \mathrm{Sel}_2(A^\chi) = \dim_{\mathbb{F}_2} \mathrm{Sel}_2(A) + \dim_{\mathbb{F}_2} V_S^\chi - \dim_{\mathbb{F}_2} V_S. \tag{4.8}$$

Moreover, we have:

$$\dim_{\mathbb{F}_2} V_S + \dim_{\mathbb{F}_2} V_S^\chi \leq \sum_{v\in S} \dim_{\mathbb{F}_2} \frac{\mathrm{im}(\delta_v)}{\mathrm{im}(\delta_v) \cap \mathrm{im}(\delta_v^\chi)}.$$

Proof. To ease notation, for each place v if k, write $\mathcal{L}_v = \delta(A(k_v))$ and $\mathcal{F}_v = \delta(A^\chi(k_v))$. Define a Selmer structure $\mathcal{G}$ for $A[2]$ by setting $\mathcal{G}_v = \mathcal{L}_v \cap \mathcal{F}_v$ for each place v. We have evident exact sequences

$$0 \to \mathrm{Sel}_{\mathcal{G}}(A[2]) \to \mathrm{Sel}_2(A) \overset{\oplus_{v\in S}\mathrm{res}_v}{\to} V_S \to 0$$

and

$$0 \to \mathrm{Sel}_{\mathcal{G}}(A[2]) \to \mathrm{Sel}_2(A^\chi) \overset{\oplus_{v\in S}\mathrm{res}_v}{\to} V_S^\chi \to 0.$$

Taking $\mathbb{F}_2$-dimensions gives Equation (4.8).

We now prove (i). For each place v, since both $\mathcal{F}_v$ and $\mathcal{L}_v$ are their own orthogonal complements with respect to the local Tate pairing, we have a non-degenerate pairing

$$\mathcal{L}_v/\mathcal{G}_v \times \mathcal{F}_v/\mathcal{G}_v \to \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}$$

given by restricting the local Tate pairing. Summing over $v \in S$ gives a non-degenerate pairing

$$\oplus_{v\in S}\mathcal{L}_v/\mathcal{G}_v \times \oplus_{v\in S}\mathcal{F}_v/\mathcal{G}_v \to \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}. \tag{4.9}$$

For each $x \in \mathrm{Sel}_2(A)$ and $y \in \mathrm{Sel}_2(A^\chi)$, reciprocity for the Brauer group of k gives

$$\sum_{v \text{ place of } k} \mathrm{inv}_v(x \cup y) = 0.$$

Further, since the Selmer conditions for A and A^χ agree for all places $v \notin S$, we have $\mathrm{inv}_v(x \cup y) = 0$ for all $v \notin S$ (since for each such place, $\mathcal{F}_v = \mathcal{L}_v$ is isotropic). Thus, we deduce that V_S and V_S^χ are orthogonal with respect to the pairing (4.9). In particular, we have

$$\dim_{\mathbb{F}_2} V_S + \dim_{\mathbb{F}_2} V_S^\chi \leq \dim_{\mathbb{F}_2} \oplus_{v\in S}\mathcal{L}_v/\mathcal{G}_v,$$

from which the result follows. □

Remark 4.10. One can strengthen Proposition 4.7 by showing that we moreover have

$$\dim_{\mathbb{F}_2} V_S + \dim_{\mathbb{F}_2} V_S^\chi \equiv \sum_{v\in S} \dim_{\mathbb{F}_2} \frac{\delta(A(k_v))}{\delta(A(k_v)) \cap \delta(A^\chi(k_v))} \pmod 2.$$

This is more subtle to prove and uses the existence of certain quadratic refinements of the local Tate pairing arising from Mumford theta groups; see [Har19, Lemma 3.27] for details. However, it can be very useful in practice, in particular if one wants to twist to increase the 2-Selmer group – it sometimes lets one deduce that $\dim_{\mathbb{F}_2} V_S^\chi > 0$ just by understanding V_S.

Our basic mechanism for twisting to reduce the dimension of the 2-Selmer group is the following.

Corollary 4.11. *Let Σ be a finite set of places of k containing all archimedean places, all places dividing 2 and all places of bad reduction for A. Suppose that $\chi : G_k \to \mu_2$ is a quadratic character that is trivial locally at all places in Σ. Suppose that χ is ramified at a single place $v \notin \Sigma$ and that the restriction map $\mathrm{res}_v : \mathrm{Sel}_2(A) \to H^1_{\mathrm{ur}}(k_v, A[2])$ is surjective. Then*

$$\dim_{\mathbb{F}_2} \mathrm{Sel}_2(A^\chi) = \dim_{\mathbb{F}_2} \mathrm{Sel}_2(A) - \dim_{\mathbb{F}_2} A(k_v)[2].$$

Proof. By Lemma 4.6(i) and our assumptions on χ, we may take $S = \{v\}$ in Proposition 4.7. Using Lemma 4.4(ii) and the proposition, we have $\dim V_S + \dim V_S^\chi \leq \dim A(k_v)[2]$. Moreover, Lemma 4.4 and the assumption that the restriction map is surjective shows that $V_S = H^1_{\mathrm{ur}}(k_v, A[2])$. Now $\dim H^1_{\mathrm{ur}}(k_v, A[2]) = \dim A(k_v)[2]$, so we deduce from the inequality that $V_S^\chi = 0$, from which the result follows. □

4.4. **Parity of ranks.** A predicted consequence of the Birch and Swinnerton-Dyer conjecture together with the predicted functional equal for the L-function of A is the parity conjecture:

$$(-1)^{\mathrm{rk}A} = w(A),$$

where $w(A)$ is the *global root number* of A. Indeed, one expects that $w(A)$ is the sign in the (conjectural) functional equation for the L-function of A, hence controls the parity of the order of vanishing of the L-function at $s = 1$. The Birch and Swinnerton–Dyer conjecture then predicts that this order of vanishing is equal to the rank of A.

By definition, $w(A) \in \{\pm 1\}$ is a product of *local root numbers*:

$$w(A) = \prod_{v \text{ place of } k} w(A/k_v).$$

This means that the parity of $\mathrm{rk}A$ should be determined by relatively simple local invariants of A, namely the local root numbers $w(A/k_v) \in \{\pm 1\}$. We remark that if A has good reduction at v then $w(A/k_v) = 1$. (For an excellent introduction to the parity conjecture in the special case of elliptic curves, see [Dok13].)

Now let χ be a quadratic character with corresponding quadratic extension F/k. We have

$$\mathrm{rk}(A/F) = \mathrm{rk}(A) + \mathrm{rk}(A^\chi).$$

In particular, the parity conjecture predicts that

$$(-1)^{\mathrm{rk}(A^\chi)} = (-1)^{\mathrm{rk}A} \prod_{w \text{ place of } F} w(A/F_w). \tag{4.12}$$

Since $w(A/F_w) = 1$ whenever w lies over a place of good reduction for A, this suggests in particular that the parity of $\mathrm{rk}(A^\chi/k)$ should depend on χ only through its restriction to k_v for the finitely many places v of k at which A has bad reduction. The content of Theorem 4.15 below is that one can prove this unconditionally for 2^∞-Selmer ranks, which we define now, hence for ranks assuming finiteness of Tate–Shafarevich groups.

Definition 4.13. As in Remark 2.10, denote by $Ш_{\mathrm{nd}}(A)$ the quotient of $Ш(A)$ by its maximal divisible subgroup. Unconditionally, one has

$$Ш(A)[2^\infty] \cong \left(\mathbb{Q}_2/\mathbb{Z}_2\right)^{n_2} \oplus Ш_{\mathrm{nd}}(A)[2^\infty]$$

for some integer $n_2 \geq 0$. The 2^∞-Selmer rank of A is $\mathrm{rk}_2(A) = \mathrm{rk}(A) + n_2$. Note in particular that $\mathrm{rk}_2(A) = \mathrm{rk}(A)$ conditional on finiteness of (the 2-part of) $Ш(A)$. (It is more standard to define $\mathrm{rk}_2(A)$ as the $\mathbb{Z}_2$-corank of the 2^∞-Selmer group $\varinjlim \mathrm{Sel}_{2^n}(A)$; this is equivalent to the more concrete definition given above.)

For a place v of k and a quadratic character $\chi : G_{k_v} \to \mu_2$, denote by $\Omega_v(\chi) \in \{\pm 1\}$ the quantity defined in [Mor19, Defintion 10.21]. It depends on A (even though we omit this dependence from the notation) but only through the base-change of A to k_v.

Theorem 4.14 ([Mor19], Theorem 10.20)**.** *Let Σ be a finite set of places of k containing all archimedean places, all places dividing 2 and all places at which A has bad reduction. Let $\chi : G_k \to \mu_2$ be a global quadratic character. Then we have*

$$(-1)^{\mathrm{rk}_2 A^\chi} = (-1)^{\mathrm{rk}_2 A} \prod_{v \in \Sigma} \Omega_v(\mathrm{res}_v(\chi)).$$

Moreover, for any place v, if χ_v is the trivial character then $\Omega_v(\chi_v) = 1$.

Remark 4.15. The fundamental exact sequence (4.5) (with $n = 2$) shows that

$$\dim_{\mathbb{F}_2} \mathrm{Sel}_2(A) = \mathrm{rk}_2(A) + \dim_{\mathbb{F}_2} A(k)[2] + \dim_{\mathbb{F}_2} Ш_{\mathrm{nd}}(A)[2]. \tag{4.16}$$

The first term, $\dim_{\mathbb{F}_2} A(k)[2]$, is constant in the quadratic twist family of A. Thus, if we are in a situation where $\dim_{\mathbb{F}_2} Ш_{\mathrm{nd}}(A^\chi)[2]$ is even for *all* quadratic twists of A (as is the case when the class c_2

is zero, for example) then we deduce that the parity of $\dim_{\mathbb{F}_2} \mathrm{Sel}_2(A^\chi)$ is determined by $\{\mathrm{res}_v(\chi)\}_{v\in\Sigma}$ also. More precisely, in this situation we have

$$(-1)^{\dim \mathrm{Sel}_2(A^\chi)} = (-1)^{\dim \mathrm{Sel}_2(A)} \prod_{v\in\Sigma} \Omega_v(\mathrm{res}_v(\chi)).$$

Later, it will be useful to know that A has a quadratic twist of odd (hence positive) rank, at least assuming finiteness of Tate–Shafarevich groups. Part (i) of the following result serves to accomplish this in many instances.

Proposition 4.17. *Suppose $v \nmid 2\infty$ is a place of k such that A has semistable reduction at v, and such that the group Φ_v of geometric connected components of the special fibre of the Néron model of A over k_v has odd order. Let χ be the unique non-trivial unramified quadratic character of G_{k_v}.*

(i) *We have*

$$\Omega_v(\chi) = (-1)^{\mathfrak{f}(A/k_v)},$$

where $\mathfrak{f}(A/k_v)$ denotes the conductor exponent of A at v.[4]

(ii) *We have $\delta(A(k_v)) = H^1_{\mathrm{ur}}(k_v, A[2]) = \delta(A^\chi(k_v))$.*

Proof. (i) This is [Mor25], Proposition 6.3.

(ii). Since Néron models commute with unramified base change, it suffices to prove this for $\delta(A(k_v))$. Since $v \nmid 2\infty$, we have

$$\dim H^1_{\mathrm{ur}}(k_v, A[2]) = \dim A(k_v)[2] = \dim \delta(A(k_v)),$$

hence it suffices to show that $H^1_{\mathrm{ur}}(k_v, A[2])$ is contained in $\delta(A(k_v))$. Equivalently, we want to show that the image of $H^1_{\mathrm{ur}}(k_v, A[2])$ in $H^1(k_v, A)$ is trivial. By [Mil06, Proposition I.3.8] we have

$$H^1_{\mathrm{ur}}(k_v, A) \cong H^1_{\mathrm{ur}}(k_v, \Phi_v).$$

Since Φ_v has odd order, it follows that $H^1_{\mathrm{ur}}(k_v, A)[2]$ is trivial, hence so is the image of $H^1_{\mathrm{ur}}(k_v, A[2])$ in $H^1(k_v, A)$. $\square$

Remark 4.18. In light of Equation (4.12), given a place v of k and a quadratic character $\chi : G_{k_v} \to \mu_2$ with associated quadratic extension F/k_v, it is reasonable to expect that $\Omega_v(\chi_v) = w(A/F)$. While this is not known in general, we remark that Part (i) of Proposition 4.17 shows that this is the case when A has semistable reduction at v, χ_v is unramified, and Φ_v has odd order. Indeed, even without the assumptions on the reduction of A at v, it is shown in [Čes18, Corollary 4.6] that one has $w(A/F) = (-1)^{\mathfrak{f}(A/k_v)}$ where F/k_v denotes the unique unramified quadratic extension of k.

5. Rational points on Kummer varieties

In this section, we discuss two results about the existence of rational points on Kummer varieties obtained via Swinnerton-Dyer's method. For concreteness, we restrict to the case of Kummer surfaces associated to Jacobians of genus 2 curves (though both results admit natural generalisations to higher dimensional Kummer varieties). We begin by making some of the concepts introduced previously more explicit in this case.

5.1. Jacobians of genus 2 curves.

Let k be a field of characteristic different from 2. Let $f(x) \in k[x]$ be a separable polynomial of degree 5 or 6, and let $C : y^2 = f(x)$ be the associated genus 2 curve. Let A be the Jacobian of C. Denote by $\mathcal{W}$ the G_k-set of Weierstrass points on C. Thus, we have

$$\mathcal{W} = \begin{cases} \{(r,0) \ : \ r \text{ is a root of } f(x)\} & \deg(f) = 6, \\ \{(r,0) \ : \ r \text{ is a root of } f(x)\} \cup \{\infty\} & \deg(f) = 5. \end{cases}$$

[4] That is, the conductor exponent associated to the ℓ-adic Tate module of A for any prime ℓ with $v \nmid \ell$; see [Ser70, Section 2] for the definition.

Here in the case where $\deg(f) = 5$, ∞ denotes the unique point on C not seen on the affine chart $y^2 = f(x)$ (when $\deg(f) = 6$ there are instead 2 such points, which are not Weierstrass points).

As a G_k-module, $A[2] \cong \mathbb{F}_2[\mathcal{W}]_{\mathrm{sum}=0}/\left\langle \sum_{w\in\mathcal{W}} w\right\rangle$ (the key point is that, for each pair P, Q of Weierstrass points, the class of the divisor $P - Q$ is 2-torsion). One sees from this description that $G = \mathrm{Gal}(k(A[2])/k)$ is naturally identified with the Galois group of $f(x)$. Moreover, both the Weil pairing on $A[2]$ and the G_k-set of quadratic refinements of the Weil pairing on $A[2]$ can be described explicitly via this isomorphism. Indeed, we can naturally think of $\mathbb{F}_2[\mathcal{W}]/\left\langle \sum_{w\in\mathcal{W}} w\right\rangle$ as the collection of even-sized subsets of $\mathcal{W}$, modulo the equivalence relation identifying a subset with its complement (the group operation is given by symmetric difference). Given even sized subsets S, T, we have

$$e_2(S, T) = (-1)^{\#S\cap T}.$$

Further, there is a Galois equivariant surjection from the collection of odd-sized subsets of $\mathcal{W}$, modulo the same equivalence relation, to the G_k-set of quadratic refinements of e_2. It is given by sending a subset T to the quadratic form $q_T : A[2] \to \mu_2$ defined by

$$q_T(S) = (-1)^{\frac{1}{2}\#S+\#S\cap T}.$$

See [Dol12, Section 5.2.3] for details.

When $\deg(f) = 5$ there is thus a G_k-invariant quadratic refinement of the Weil pairing on $A[2]$ associated to the point at infinity.

With some work, one can use the discussion above to prove the following:

Lemma 5.1. *Suppose that* $\mathrm{Gal}(f) = S_{\deg(f)}$. *Then* $A[2]$ *is a simple* G*-module and* $\mathrm{End}_G(A[2]) = \mathbb{F}_2$. *If* $\deg(f) = 5$ *then* $H^1(G, A[2]) = 0$ *and (in particular)* $c_2 = 0$. *If* $\deg(f) = 6$ *then* $H^1(G, A[2])$ *is* 1*-dimensional, generated by the class* c_2 *parameterising quadratic refinements of the Weil pairing.*

5.2. **Extensions defined by cocyles.** We return to the general situation where A/k is a principally polarised abelian variety over a field k of characteristic different from 2. Write $G = \mathrm{Gal}(k(A[2])/k)$. Let $T = \{\alpha_1, ..., \alpha_t\}$ be a finite subset of $H^1(k, A[2])$ and represent each α_i by a 1-cocycle $\tilde{\alpha}_i : G_k \to A[2]$. Then the map $\varphi_T : G_k \to A[2]^T \rtimes G$, given by

$$\sigma \mapsto \big(\tilde{\alpha}_1(\sigma), \ldots, \tilde{\alpha}_t(\sigma), \bar{\sigma}\big),$$

is a homomorphism. Here $\bar{\sigma}$ denotes the image of σ in G. Denote by k_T the fixed field of the kernel of this homomorphism, so that we have an injection $\varphi_T \colon \mathrm{Gal}(k_T/k) \hookrightarrow A[2]^T \rtimes G$. The extension k_T/k contains $k(A[2])/k$ and is the smallest Galois extension of k through which each of the cocycles $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_t$ factor.

Consider the following condition:

Condition 5.2. We have

(a) $H^1(G, A[2]) = 0$,

(b) $A[2]$ is a simple $\mathbb{F}_2[G]$-module and $\mathrm{End}_G(A[2]) = \mathbb{F}_2$.

Lemma 5.3. *Suppose that Condition 5.2 is satisfied. Suppose also that* T *is* $\mathbb{F}_2$*-linearly independent.*

(i) $\varphi_T : \mathrm{Gal}(k_T/k) \to A[2]^T \rtimes G$ *is an isomorphism.*

(ii) *Let* k_T^{ab} *denote the maximal abelian subextension of* k_T/k. *Then* $k_T^{\mathrm{ab}} \subseteq k(A[2])$.

Proof. (i) It suffices to show that the restriction of φ_T to $L = k(A[2])$ induces a surjection $\mathrm{Gal}(k_T/L) \to A[2]^T$. Since $H^1(G, A[2]) = 0$, the inflation-restriction exact sequence shows that $\{\alpha_1|_L, ..., \alpha_t|_L\}$ is a linearly independent subset of $\mathrm{Hom}_G\big(\mathrm{Gal}(k_T/L), A[2]\big)$, where G acts on $\mathrm{Gal}(k_T/L)$ by conjugation. Note in particular that $\dim_{\mathbb{F}_2} \mathrm{Hom}_G(\mathrm{Gal}(k_T/L), A[2]) \geq t$.

Let $M = \varphi_T(\mathrm{Gal}(k_T/L))$. Then M is an $\mathbb{F}_2[G]$-submodule of $A[2]^T$. Since $A[2]$ is a simple $\mathbb{F}_2[G]$-module, and $M \leq A[2]^T$, we have $M \cong A[2]^r$ for some $r \leq t$. On the other hand, using Condition 5.2 (b) we compute

$$\dim_{\mathbb{F}_2} \mathrm{Hom}_G(\mathrm{Gal}(k_T/L), A[2]) = \dim_{\mathbb{F}_2} \mathrm{Hom}_G(M, A[2]) = \dim_{\mathbb{F}_2} \mathrm{Hom}_G(A[2]^r, A[2]) = r.$$

Comparing with the first paragraph we obtain $r \geq t$. We thus conclude that $r = t$, hence $M = A[2]^T$.

(ii) Let M be any abelian group and let $\theta : A[2]^T \rtimes G \to M$ be a homomorphism. In light of (i), we want to show that $A[2]^T \subseteq \ker(\theta)$, so that then θ factors through $G = \mathrm{Gal}(k(A[2])/k)$. Consider the restriction of θ to one of the factors $A[2]$ of $A[2]^T$. Since M is abelian, and since the action of G on $A[2]^T$ is given by (or rather, can be realised by) lifting to the semidirect product and conjugating, we see that the restriction of θ to a homomorphism $A[2] \to M$ is G-equivariant, where M is given the trivial action. The kernel of this map is then either trivial or all of $A[2]$. But the former can't happen since it would realise $A[2]$ (which has non-trivial action) as a submodule of M (which has trivial action). Thus the restriction of θ to each factor of $A[2]^T$ is trivial, which is enough to conclude the result. □

5.3. **Quadratic characters with specified local behaviour.** In this subsection, we state and prove a lemma that will help us to accomplish Step (1) of the general strategy set out in Section 3.2.

Fix a finite Galois extension F/k of number fields with Galois group Γ. Fix also a finite set Σ of places of k containing all places dividing 2∞, and all places ramified in F/k.

Notation 5.4. Let $\mathcal{C} \subseteq \Gamma$ be a non-empty union of conjugacy classes and write $\mathcal{P}(\mathcal{C})$ for the set of primes $\mathfrak{p} \notin \Sigma$ for which the Frobenius element $\mathrm{Frob}_{\mathfrak{p}} \in \Gamma$ lies in $\mathcal{C}$. Let $\mathcal{A}(\mathcal{C})$ denote the set of quadratic characters $\chi : \Gamma \to \{\pm 1\}$ satisfying $\chi(\sigma) = 1$ for all $\sigma \in \mathcal{C}$.

Lemma 5.5. *Let $(\chi_v)_{v\in\Sigma}$ be a collection of quadratic characters $\chi_v : G_{k_v} \to \mu_2$. Assume that, for every $\psi \in \mathcal{A}(\mathcal{C})$, we have*

$$\sum_{v\in\Sigma} \mathrm{inv}_v(\chi_v \cup \mathrm{res}_v(\psi)) = 0.$$

Then there is a global quadratic character $\chi : G_k \to \mu_2$ such that $\mathrm{res}_v(\chi) = \chi_v$ for all $v \in \Sigma$, and such that χ is unramified outside $\Sigma \cup \mathcal{P}(\mathcal{C})$.

Proof. Exactness at the middle term of the Poitou–Tate exact sequence (see, for example, [Mil06, Theorem I.4.10]) applied to the set $\Sigma \cup \mathcal{P}(\mathcal{C})$ of places, and to the self-dual G_k-module μ_2, shows that

$$\mathrm{im}\Big(H^1(k_{\Sigma\cup\mathcal{P}(\mathcal{C})}/k, \mu_2) \xrightarrow{\mathrm{res}} {\prod_{v\in\Sigma\cup\mathcal{P}(\mathcal{C})}}' H^1(k_v, \mu_2)\Big)$$

is its own orthogonal complement under the sum of the local Tate pairings. Here $k_{\Sigma\cup\mathcal{P}(\mathcal{C})}$ denotes the maximal extension of k unramified outside $\Sigma \cup \mathcal{P}(\mathcal{C})$, and the restricted direct product is taken with respect to the subgroups of unramified classes. Projecting onto $\prod_{v\in\Sigma} H^1(k_v, \mu_2)$, it follows formally that the image of $H^1(k_{\Sigma\cup\mathcal{P}(\mathcal{C})}/k, \mu_2)$ in $\prod_{v\in\Sigma} H^1(k_v, \mu_2)$ is the orthogonal complement of the image of

$$\ker\Big(H^1(k_\Sigma/k, \mu_2) \xrightarrow{\mathrm{res}} {\prod_{v\in\mathcal{P}(\mathcal{C})}}' H^1(k_v, \mu_2)\Big) \tag{5.6}$$

in $\prod_{v\in\Sigma} H^1(k_v, \mu_2)$. Now let $\psi \in H^1(k_\Sigma/k, \mu_2)$. If either ψ does not factor through Γ, of if there is $\sigma \in \mathcal{C}$ with $\psi(\sigma) = -1$, then by the Chebotarev density theorem we can find a prime $\mathfrak{p} \in \mathcal{P}(\mathcal{C})$ such that the restriction of χ to $G_{k_{\mathfrak{p}}}$ is non-trivial. From this we conclude that the kernel in (5.6) is equal to $\mathcal{A}(\mathcal{C})$, giving the result. □

5.4. **Hasse principle for Kummer surfaces.** The following is a result of Harpaz–Skorobogatov [HS16, Theorem B] (see [HS16, Theorem 2.3] for a higher dimensional generalisation). We give a slight variant of their argument, incorporating results on the parity of ranks to enable one to twist only by characters that are trivial at all places of bad reduction.

Theorem 5.7 (Harpaz–Skorobogatov)**.** *Let $f(x) \in \mathcal{O}_k[x]$ be a separable monic polynomial of degree 5 and let $C : y^2 = f(x)$ be the corresponding genus 2 curve. Let A be the Jacobian of C. Suppose that $\mathrm{Gal}(f) \cong S_5$ and that there is a odd prime ideal $\mathfrak{p}$ with $\mathrm{ord}_{\mathfrak{p}}\mathrm{disc}(f) = 1$. Let $\alpha \in H^1(k, A[2])$ and* *suppose that α is unramified at $\mathfrak{p}$. Assume that the Tate–Shafarevich group of each quadratic twist of A is finite. Then the Kummer surface X_α satisfies the Hasse principle.*

We begin with a lemma. The key input in its proof is Lemma 5.5. We remark that this lemma can be replaced by an application of the fibration method. This approach relates the existence of a character as in the statement to the vanishing of a suitable vertical Brauer group and gives a more geometric approach to this part of the argument. See [HS16, Proposition 6.2] and the references therein for details.

Lemma 5.8. *Take all the notation and assumptions of Theorem 5.7. Suppose that $\alpha \neq 0$ and $X_\alpha(\mathbb{A}_k) \neq \emptyset$. Then there is a quadratic character $\chi_0 : G_k \to \mu_2$ such that $\alpha \in \mathrm{Sel}_2(A^{\chi_0})$ and $\dim \mathrm{Sel}_2(A^{\chi_0}) \equiv 1 \pmod 2$.*

Proof. Let Σ be a finite set of places containing all places dividing 2, all archimedean places and all places of bad reduction for A. The assumption that $X_\alpha(\mathbb{A}_k) \neq \emptyset$ means that for each place $v \in \Sigma \setminus \{\mathfrak{p}\}$ there is a local quadratic character $\chi_v : G_{k_v} \to \mu_2$ such that $Y_\alpha^{\chi_v}(k_v) \neq \emptyset$. That is, such that $\mathrm{res}_v(\alpha) \in \mathrm{im}(\delta_v^{\chi_v})$.

The assumptions on the special prime $\mathfrak{p}$ entail that the geometric special fibre of the Néron model of A over $k_\mathfrak{p}$ is connected and that $\mathfrak{f}(A/k_\mathfrak{p}) = 1$ (cf. [HS16, Proof of Corollary 2.4]). Let $\psi_\mathfrak{p}$ be the unique non-trivial unramified quadratic character. Since α is unramified at $\mathfrak{p}$, it follows from Proposition 4.17(ii) that α lies in both $\mathrm{im}(\delta_\mathfrak{p})$ and $\mathrm{im}(\delta_\mathfrak{p}^{\psi_\mathfrak{p}})$. Since $\mathfrak{f}(A/k_\mathfrak{p})$ is odd, by Proposition 4.17(i), Theorem 4.14 and Remark 4.15 we can, choosing $\chi_\mathfrak{p}$ to be either the trivial character or non-trivial unramified character as appropriate, ensure that $\dim \mathrm{Sel}_2(A^\chi)$ is odd for any global character $\chi : G_k \to \mu_2$ such that $\mathrm{res}_v(\chi) = \chi_v$ for all $v \in \Sigma$. For any such χ, we further have $\mathrm{res}_v(\alpha) \in \mathrm{im}(\delta_v^\chi)$ for each $v \in \Sigma$.

Let $T = \{\alpha\}$ and let $\widetilde{\alpha}$ be a 1-cocycle representing α. Lemma 5.1 and Lemma 5.3 together show that the map $\varphi_T : \mathrm{Gal}(k_T/k) \to A[2] \rtimes \mathrm{Gal}(f)$, given by $\sigma \mapsto (\widetilde{\alpha}, \bar{\sigma})$, is an isomorphism. We also deduce from Lemma 5.3 that the maximal abelian subextension of k_T/k is $k(\sqrt{\mathrm{disc}(f)})$. Let $\mathcal{C} \subseteq \mathrm{Gal}(k_T/k)$ be the set of elements conjugate to an element of $\varphi_T^{-1}(0 \times \mathrm{Gal}(f))$. Since this contains an element acting on the roots of $f(x)$ as an odd permutation, the set $\mathcal{A}(\mathcal{C})$ in Lemma 5.5 is trivial. We deduce the existence of a global quadratic character χ with $\mathrm{res}_v(\chi) = \chi_v$ for all $v \in \Sigma$ and such that, outside Σ, χ is ramified only at primes $\mathfrak{q}$ for which $\mathrm{Frob}_\mathfrak{q} \in \mathcal{C}$. By construction, $\mathrm{res}_\mathfrak{q}(\alpha) = 0$ for all such primes. This, combined with the choice of $\{\chi_v\}_{v\in\Sigma}$, ensures that $\alpha \in \mathrm{Sel}_2(A^\chi)$, completing the proof. □

Proof of Theorem 5.7. Without loss of generality, we can take $\alpha \neq 0$ (else X_α trivially has a k-point). Further, we can assume $X_\alpha(\mathbb{A}_k) \neq \emptyset$ (else X_α trivially satisfies the Hasse principle).

Since $\deg(f) = 5$ the class c_2 parameterising quadratic refinements of the Weil pairing is zero (by the discussion preceding Lemma 5.1) . Thus, conditional on finiteness of Tate–Shafarevich groups, we have $\dim Ш(A^\chi)[2] \equiv 0 \pmod 2$ for all quadratic characters χ (cf. Proposition 2.11).

Let $\chi_0 : G_k \to \mu_2$ be a character afforded by Lemma 5.8, so that $\alpha \in \mathrm{Sel}_2(A^{\chi_0})$ and $\dim \mathrm{Sel}_2(A^{\chi_0}) \equiv 1 \pmod 2$. Fix a finite set Σ of places of k containing all places dividing 2, all archimedean places, all places of bad reduction for A and all places where χ_0 ramifies.

Claim. Suppose that $\chi : G_k \to \mu_2$ is a quadratic character such that $\alpha \in \mathrm{Sel}_2(A^\chi)$ and $\dim \mathrm{Sel}_2(A^\chi) \geq 3$. Then there is a quadratic character $\chi' : G_k \to \mu_2$ such that $\alpha \in \mathrm{Sel}_2(A^{\chi'})$ and $\dim \mathrm{Sel}_2(A^{\chi'}) = \dim \mathrm{Sel}_2(A^\chi) - 2$.

Proof of claim: To ease notation, take $\alpha_1 := \alpha$. Extend α_1 to a basis $T = \{\alpha_1, ..., \alpha_t\}$ for $\mathrm{Sel}_2(A^\chi)$. As above, the assumptions of Lemma 5.3 are satisfied. Thus, representing each α_i by a 1-cocycle $\widetilde{\alpha}_i$, the resulting homomorphism $\varphi_T : \mathrm{Gal}(k_T/k) \to A[2]^t \rtimes \mathrm{Gal}(f)$, given by

$$\sigma \mapsto (\widetilde{\alpha}_1(\sigma), ..., \widetilde{\alpha}_t(\sigma), \bar{\sigma})$$

is an isomorphism (here, as before, $\bar{\sigma}$ denotes the image of σ in $\mathrm{Gal}(f)$).

Let $\widetilde{\Sigma} \supseteq \Sigma$ be a finite set of places containing all places where χ ramifies. Note that k_T/k is unramified outside $\widetilde{\Sigma}$ (as follows from Lemma 4.6(i) and the fact that $\alpha_1, ..., \alpha_t \in \mathrm{Sel}_2(A^\chi)$). Let $\mathfrak{m}$ be the formal product of 8 and all places in $\widetilde{\Sigma}$, and let $k_\mathfrak{m}/k$ denote the ray class field of conductor $\mathfrak{m}$.

Note that $t = \dim \mathrm{Sel}_2(A^\chi) \geq 3$ by assumption. Note also that, for every double transposition $\tau \in \mathrm{Gal}(f)$, we have $\dim_{\mathbb{F}_2} A[2]^\tau = \dim_{\mathbb{F}_2} A[2]/(\tau - 1)A[2]$ (cf. Section 5.1). We now fix $\sigma \in \mathrm{Gal}(k_\mathfrak{m} k_T/k)$ such that:

- $\sigma|_{k_\mathfrak{m}} = 1$,
- $\varphi_T(\sigma) = (0, x_2, x_3, 0, ..., 0, \tau)$ where $\tau \in \mathrm{Gal}(f)$ is a double transposition and $\{x_2, x_3\}$ is an $\mathbb{F}_2$-basis for $A[2]/(\tau - 1)A[2]$.

To see that such an element exists we are using: that φ_T is an isomorphism, that $k_\mathfrak{m}/k$ is abelian, that the maximal abelian subextension of k_T/k is $k(\sqrt{\mathrm{disc}(f)})/k$ (by Lemma 5.3 and the assumption $\mathrm{Gal}(f) \cong S_5$), and that $\tau \in A_5$ necessarily acts trivially on $k(\sqrt{\mathrm{disc}(f)}) = k_T \cap k_\mathfrak{m}$.

Now choose a prime ideal $\mathfrak{q} \notin \widetilde{\Sigma}$ such that the Frobenius element $\mathrm{Frob}_\mathfrak{q} \in \mathrm{Gal}(k_T k_\mathfrak{m}/k)$ lies in the conjugacy class of σ. Since $\mathrm{Frob}_\mathfrak{q}$ is trivial in $\mathrm{Gal}(k_\mathfrak{m}/k)$, the prime ideal $\mathfrak{q}$ is principal, generated by an element $\pi \in \mathcal{O}_k$ that is a square locally at all places $v \in \widetilde{\Sigma}$.[5] Let χ_π be the quadratic character associated to $k(\sqrt{\pi})/k$ and let $\chi' = \chi\chi_\pi$. By construction, χ_π is trivial at all places in $\widetilde{\Sigma}$ and is ramified at the single place v corresponding to $\mathfrak{q}$. Further, the composition

$$\mathrm{Sel}_2(A^\chi) \xrightarrow{\mathrm{res}_\mathfrak{q}} H^1_{\mathrm{ur}}(k_\mathfrak{q}, A[2]) \cong A[2]/(\mathrm{Frob}_\mathfrak{q} - 1)A[2],$$

which evaluates cocycles at $\mathrm{Frob}_\mathfrak{q}$, is surjective thanks to our choice of $\mathrm{Frob}_\mathfrak{q} \in \mathrm{Gal}(k_T/k)$ (in fact, the images of α_2 and α_3 span $A[2]/(\mathrm{Frob}_\mathfrak{q} - 1)A[2]$). We may thus apply Corollary 4.11 to the abelian variety A^χ and the quadratic character χ_π, giving

$$\dim \mathrm{Sel}_2(A^{\chi'}) = \dim \mathrm{Sel}_2(A^\chi) - \dim A(k_\mathfrak{q})[2] = \dim \mathrm{Sel}_2(A^\chi) - 2,$$

the final equality following from the choice of τ. Finally, since we chose σ such that $\widetilde{\alpha}(\sigma) = 0$, we see that $\mathrm{res}_\mathfrak{q}(\alpha) = 0$. Thus α satisfies the Selmer conditions for $A^{\chi'}$ at $\mathfrak{q}$. Since the Selmer conditions for A^χ and $A^{\chi'}$ agree away from $\mathfrak{q}$, we deduce that $\alpha \in \mathrm{Sel}_2(A^{\chi'})$, completing the proof of the claim.

To complete the proof of the theorem we apply the claim inductively, starting from $\chi = \chi_0$. Having arranged that $\dim \mathrm{Sel}_2(A^{\chi_0})$ is odd, we eventually arrive at a quadratic character χ' for which $\mathrm{Sel}_2(A^{\chi'})$ is one dimensional, generated by α. As remarked at the start of the proof, the fact that $\deg(f) = 5$ ensures that $c_2 = 0$ hence, conditional on finiteness of $\text{Ш}(A^{\chi'})$, we have $\dim \text{Ш}(A^{\chi'})[2] \equiv 0 \pmod 2$. As explained in Section 3.2, this is enough to conclude that $X_\alpha(k) \neq \emptyset$. □

Remark 5.9. The proof of this result in [HS16, Theorem B] is slightly different. Specifically, they do not initially arrange that $\dim \mathrm{Sel}_2(A^{\chi_0})$ is odd (removing the need for Theorem 4.14 in this case). Rather, they twist inductively by characters ramified at a single prime whose Frobenius element fixes a 1-dimensional subspace of $A[2]$ (e.g. a 4-cycle) in order to reduce the dimension of the 2-Selmer group by 1 at each step. However, by parity considerations, this is not possible to do with characters that are trivial locally at all places of bad reduction for A, all places over 2 and all archimedean places; their characters all restrict to the non-trivial unramified character locally at the distinguished place $\mathfrak{p}$. One advantage of using double transpositions instead of 4-cycles is that one can prove instances of Theorem 5.7 for other Galois groups such as A_5 (in fact, all transitive subgroups of S_5 save C_5; see [MS24, Section 4]).[6]

We now briefly discuss the corresponding result where $\deg(f) = 6$, which is a special case of [Mor25, Theorem 1.9].

[5]Indeed, recall from class field theory that the map sending a prime $\mathfrak{q} \nmid \mathfrak{m}$ to $\mathrm{Frob}_\mathfrak{q}$ induces an isomorphism $I(\mathfrak{m})/P(\mathfrak{m}) \cong \mathrm{Gal}(k_\mathfrak{m}/k)$, where $I(\mathfrak{m})$ denotes the group of fractional ideals of k coprime to $\mathfrak{m}$ and $P(\mathfrak{m})$ denotes the subgroup of principal fractional ideals generated by an element $x \in k^\times$ satisfying $v_\mathfrak{l}(x-1) \geq v_\mathfrak{l}(\mathfrak{m})$ for all primes $\mathfrak{l} \mid \mathfrak{m}$ and $\sigma(x) > 0$ for each real embedding $\sigma \mid \mathfrak{m}$.

[6]The condition at the special prime $\mathfrak{p}$ must be slightly modified for other groups since the current version precludes the discriminant from being a square.

Theorem 5.10. *Let $f(x) \in \mathcal{O}_k[x]$ be a separable monic polynomial of degree 6 and let $C : y^2 = f(x)$ be the corresponding genus 2 curve. Let A be the Jacobian of C. Suppose that $\mathrm{Gal}(f) \cong S_6$ and that there is a prime ideal $\mathfrak{p}$ with $\mathrm{ord}_{\mathfrak{p}}\mathrm{disc}(f) = 1$. Let $\alpha \in H^1(k, A[2])$ and suppose that α is unramified at $\mathfrak{p}$. Assume that the Tate–Shafarevich group of each quadratic twist of A is finite. Then the Kummer surface X_α satisfies the Hasse principle.*

Unlike the situation of Theorem 5.7, the class $c_2 \in H^1(k, A[2])$ parameterising quadratic refinements of the Weil pairing is now non-trivial. Since this is the class measuring the obstruction to the Cassels–Tate pairing being alternating, for a quadratic character χ it is no longer possible to conclude that $Ш(A^\chi)[2] = 0$ from the fact that $\dim \mathrm{Sel}^2(A^\chi) = 1$ alone. Moreover, one has $c_2 \in \mathrm{Sel}^2(A^\chi)$ for *all* quadratic characters χ, so the most one could hope to obtain from the Mazur–Rubin twisting procedure is the existence of a final character χ_n such that $\mathrm{Sel}^2(A^{\chi_n})$ is generated by α and c_2.

To overcome these issues, one can incorporate second descent ideas in the spirit of Harpaz [Har19] and Smith [Smi16], and study the variation of the Cassels–Tate pairing on the 2-Selmer group under quadratic twist. Specifically, for a quadratic character χ, we can pull back the Cassels–Tate pairing along the surjection $\mathrm{Sel}_2(A^\chi) \to Ш(A^\chi)[2]$ to give a pairing

$$\mathrm{CTP}_\chi : \mathrm{Sel}_2(A^\chi) \times \mathrm{Sel}_2(A^\chi) \to \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}.$$

One can show that its kernel is $2\mathrm{Sel}_4(A^\chi)$, where $\mathrm{Sel}_4(A^\chi)$ denotes the 4-Selmer group of A^χ. Note that any class in $\mathrm{Sel}_2(A)$ in the image of $\delta : A(k)/2A(k) \to \mathrm{Sel}_2(A)$ is necessarily in the kernel of CTP_χ.

The strategy is as follows:[7]

1. similarly to the degree 5 case, under the assumptions of the theorem, local solubility of the Kummer variety X_α implies the existence of a quadratic character χ_0 such that α lies in $\mathrm{Sel}_2(A^{\chi_0})$ and $\mathrm{rk}_2(A^{\chi_0})$ is odd (and in particular positive),
2. instead of twisting to shrink the 2-Selmer group, use the Mazur–Rubin machinery to produce many quadratic twists all having odd 2^∞-Selmer rank and identical 2-Selmer group to that of A^{χ_0},
3. as one varies over twists as in the previous step, show that the resulting Cassels–Tate pairings on the common 2-Selmer group vary over all possible pairings subject to certain forced structure,
4. use the previous step to find a twist χ for which α generates the kernel of the corresponding Cassels–Tate pairing. To conclude from this, note that α is then the unique non-zero element lifting to the 4-Selmer group of A^χ. Since A^χ has odd 2^∞-Selmer rank by construction, the only possibility, assuming finiteness of $Ш(A^\chi/K)[2^\infty]$, is that α is in the image of $\delta : A(k)/2A(k) \to H^1(k, A[2])$, hence has trivial image in $Ш(A^\chi)$. Thus $X_\alpha(k) \neq \emptyset$.

The second step uses the existence of elements $\sigma \in \mathrm{Gal}(f)$ such that $A[2]^\sigma = 0$, e.g. 5-cycles. The key point is that, if $\mathfrak{q}$ is a prime such that $A[2]^{\mathrm{Frob}_\mathfrak{q}} = 0$, then $H^1(k_\mathfrak{q}, A[2]) = 0$ hence the Selmer conditions at $\mathfrak{q}$ are vacuous; for this reason, such primes are referred to as 'silent primes' in the recent work of Alpöge–Bhagava–Ho–Shnidman [ABHS26].

Regarding the third step, the following general result is [Mor25, Theorem 1.14]. To state it, let A/k be a principally polarised abelian variety and let $G = \mathrm{Gal}(k(A[2])/k)$.

Condition 5.11. Suppose that:

(A) $A[2]$ is a simple G-module and $\mathrm{End}_G(A[2]) = \mathbb{F}_2$,
(B) there is $g \in G$ with $A[2]^g = 0$,
(C) $H^1(G, A[2]) \cap \mathrm{Sel}_2(A) = \langle c_2 \rangle$ (we allow for the possibility that $c_2 = 0$).

This condition holds, for example, if G is the full symplectic group of automorphisms of $A[2]$ preserving the Weil pairing.

[7] One could try to first twist to make the 2-Selmer group as small as possible before moving on to the the study of the Cassels–Tate pairing, but this turns out not to be necessary for the argument.

Definition 5.12. Call a $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$-valued bilinear pairing P on $\mathrm{Sel}_2(A)$ *admissible* if: $P(x,x) = P(x,c_2)$ for all $x \in \mathrm{Sel}_2(A)$, and

$$P(c_2,c_2) = \begin{cases} 0 & \dim \mathrm{Sel}_2(A) + \mathrm{rk}_2(A) \equiv 0 \pmod 2, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

It follows from the work of Poonen–Stoll that the Cassels–Tate pairing on $\mathrm{Sel}_2(A)$ is admissible. (If the class c_2 is zero, then P is admissible if and only if P is alternating.)

Theorem 5.13 ([Mor25], Theorem 1.14)**.** *Assume that Condition 5.11 holds. Let P be any admissible pairing on $\mathrm{Sel}_2(A)$. Then there is a quadratic character $\chi : G_K \to \mu_2$ such that all of the following hold:*

(i) $\mathrm{rk}_2(A^\chi) \equiv \mathrm{rk}_2(A) \pmod 2$,
(ii) $\mathrm{Sel}_2(A^\chi) = \mathrm{Sel}_2(A)$ *as subgroups of* $H^1(k, A[2])$,
(iii) *for all* $x, y \in \mathrm{Sel}_2(A^\chi)$, *we have*

$$\mathrm{CTP}_\chi(x,y) = P(x,y).$$

We conclude these notes by explaining how to apply this result to carry out the fourth step of the strategy for proving Theorem 5.10 outlined above. Suppose we are in the situation of Theorem 5.10. We can suppose that $\alpha \neq 0$ since otherwise X_α trivially has a k-point. We can also suppose that $c_2 \neq \alpha$. Indeed, X_{c_2} is the minimal desingularisation of the quotient of $\mathrm{Pic}^1_{C/k}$ by the involution induced from the hyperelliptic involution ι on C. The natural embedding of C into $\mathrm{Pic}^1_{C/k}$ induces an embedding $C/\iota \cong \mathbb{P}^1 \to X_{c_2}$. In particular $X_{c_2}(k) \neq \emptyset$.[8]

Assume now that $c_2 \neq \alpha \neq 0$ and that X_α is everywhere locally soluble. Suppose we are given a character χ_0 as in the first step of the strategy, so that $\alpha \in \mathrm{Sel}_2(A^{\chi_0})$ and $\mathrm{rk}_2(A^{\chi_0})$ is odd. Since $\mathrm{Gal}(f) \cong S_6$, it follows from Lemma 5.1 that c_2 is non-zero, so that α and c_2 span a 2-dimensional subspace of $\mathrm{Sel}_2(A^{\chi_0})$. The same lemma also shows that Condition 5.11 holds for A^{χ_0}. Thus, Theorem 5.13 applies to A^{χ_0}. To complete the fourth step of the strategy, it suffices to demonstrate the existence of an admissible pairing P on $\mathrm{Sel}_2(A^{\chi_0})$ whose kernel is generated by α.

Case 1: suppose first that $\dim \mathrm{Sel}_2(A^{\chi_0}) \equiv 0 \pmod 2$. Since $\mathrm{rk}_2(A^{\chi_0})$ is odd, a pairing P on $\mathrm{Sel}_2(A^{\chi_0})$ is admissible if and only if $P(c_2,c_2) = \frac{1}{2}$ and $P(x,x) = P(x,c_2)$ for all $x \in \mathrm{Sel}_2(A^{\chi_0})$. We extend α, c_2 to a basis $\alpha, c_2, x_1, ..., x_n$ for $\mathrm{Sel}_2(A^{\chi_0})$, noting that n is even. Take P to be the pairing on $\mathrm{Sel}_2(A^{\chi_0})$ whose matrix with respect to this basis is

$$\begin{pmatrix} 0 & 0 & & & & & \\ 0 & \frac{1}{2} & & & & & \\ & & 0 & \frac{1}{2} & & & \\ & & \frac{1}{2} & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \frac{1}{2} \\ & & & & & \frac{1}{2} & 0 \end{pmatrix}.$$

We have $P(c_2,c_2) = \frac{1}{2}$. Moreover, since P is symmetric, the identity $P(x,x) = P(x,c_2)$ holds for all $x \in \mathrm{Sel}_2(A^{\chi_0})$ if and only if it does so as x varies over a basis. Since it holds for the basis $\alpha, c_2, x_1, ..., x_n$ by construction, we see that P is admissible. Finally, we note that the displayed matrix has $\mathbb{F}_2$-rank $\dim \mathrm{Sel}_2(A^{\chi_0}) - 1$ and contains α in its kernel.

Case 2: now suppose that $\dim \mathrm{Sel}_2(A^{\chi_0}) \equiv 1 \pmod 2$. This time, since we still have $\mathrm{rk}_2(A^{\chi_0})$ odd, a pairing P on $\mathrm{Sel}_2(A^{\chi_0})$ is admissible if and only if $P(c_2,c_2) = 0$ and $P(x,x) = P(x,c_2)$ for all

[8] More than this, it is a curious fact that (in the specific case of Jacobians of genus 2 curves) the Kummer varieties associated to c_2 and to the zero class are isomorphic; see [GCH22, Section 3.2] for a discussion of this phenomenon.

$x \in \mathrm{Sel}_2(A^{\chi_0})$. As before, we extend α, c_2 to a basis $\alpha, c_2, x_1, ..., x_n$ for $\mathrm{Sel}_2(A^{\chi_0})$, noting that now n is odd. Take P to be the pairing on $\mathrm{Sel}_2(A^{\chi_0})$ whose matrix with respect to this basis is

$$\begin{pmatrix} 0 & & & & & \\ & 0 & \frac{1}{2} & & & \\ & \frac{1}{2} & \frac{1}{2} & & & \\ & & & \ddots & & \\ & & & & 0 & \frac{1}{2} \\ & & & & \frac{1}{2} & 0 \end{pmatrix}.$$

As in the previous case, P is an admissible pairing whose kernel is generated by α.

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