Solvable Descent and the Grunwald Problem for Solvable Groups

Lectures by Julian Demeio

Schedule of lectures: Monday 30th March, Wednesday 1st April, Friday 3rd April, 11 am to 1 pm.

Content:

The aim of the course is proving "Solvable Descent". This gives as a corollary a positive solution to the Grunwald Problem for solvable groups (hence the title).

Using a descent technique developed by Harpaz and Wittenberg, solvable descent reduces to the following:

Theorem ("Fibration Theorem") *Given a smooth proper fibration $f : X \to Q$ with $Q = R_{E/K}G_m$ a quasi-trivial K-torus, and for which there exists an epimorphism of K-tori $T \to Q$ such that X has a T-action under which f is T-equivariant, we have*

$$X(K_\Omega)^{\mathrm{Br}_{ur} X} = \overline{\bigcup_{q \in Q(K)} X_q(\mathbb{A}_K)}^{\mathrm{Br}(X_q)}.$$

Most of the course will be focused on the proof of this fibration theorem.

The program is roughly as follows.

Lecture 1:

- Deduction, via the descent method of Harpaz and Wittenberg, of solvable descent from the fibration theorem, and application to the Grunwald Problem.
- Introduction of the "horizontal Brauer group" $\mathrm{Br}_{hor}(X/Q)$, which uniformely parametrizes the Brauer groups of (a Hilbertian set of) fibers of f. Introduction of a natural filtration $F_i \mathrm{Br}_{hor}(X/Q), i = 0, .., 3$ in four pieces of this group.
- Construction of a toric multisection of f, which will later be used to produce local points on the fibers at bad places.
- Idea of the proof of the Fibration Theorem.
- Introduction of (generalized) Redéi symbols $[a, b, c]_n$ and discussion of their basic properties.
- Introduction of half-spin symbols. Given a number field E with an involution σ, these are variants of the "spin symbols" first introduced by Friedlander–Iwaniec in their "$p = a^4 + b^2$" paper.

Lecture 2:

- Arithmetic approximation theorem producing square-free (outside some S) algebraic numbers x in a fixed number field E satisfying: all of the prime divisors of x not in S lift in some fixed finite extension L/E; x reduces to an n-th power modulo all non-trivial conjugates of prime divisors not in S of x; all half-spin symbols of x vanish. We also show that one can find multiple such x in the shape of a grid. Additionaly one can customize the Redéi symbols associated to the variations of this grid.

The proof of the approximation theorem is inspired by the ideas of Shafarevich in his solution of the Inverse Galois Problem for solvable groups.

- Description of how the approximation theorem can be combined with the toric multi-section to produce adelic points on the fibers of f.
- When Q is of the shape $R_{E/K}G_m$ with E a field that is Galois over K, description of the Brauer—Manin pairing on the fibers for horizontal Brauer classes in $F_1 \operatorname{Br}_{\text{hor}}$ in terms of half-spin symbols.

Lecture 3:

- "Triple variation" theorem expressing the triple variation of the Brauer–Manin pairing associated to the adelic points previously introduced in terms of Redéi symbols on the base. The proof involves a computation with Čech cochains.
- Proof of the Fibration theorem (reductions): reduction of the fibration theorem to the case where $F_2 \operatorname{Br}_{\text{hor}} = 0$, E is a field that is Galois over K, and the multi-section satisfies some technical assumptions.
- Proof of the Fibration theorem (final argument): After the reductions above, the main remaining issue is Brauer—Manin orthogonality to F_3/F_2. The triple variation of the BM pairing associated to elements of this quotient is customizable (since the Redéi symbols are) on grids. A combinatorial lemma inspired by work of Alexander Smith then allows one to find a customization for which at least one fiber is forced to have trivial Brauer–Manin obstruction.