RATIONAL POINTS, ALGEBRAIC CYCLES AND THE LOCAL-GLOBAL PRINCIPLE AT LMSI: INAUGURAL LECTURE

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Abstract. These notes record a one-hour lecture delivered on January 12, 2026, during the thematic program on Rational Points, Algebraic Cycles and the Local-Global Principle, held from January to April 2026 at the Lodha Mathematical Sciences Institute in Mumbai. We survey several themes addressed during the program, including the work of Tian and Kollár-Tian establishing Colliot-Thélène's conjecture on the local-global principle for zero-cycles on geometrically rational surfaces over global function fields, the proof by Bouthier–Česnavičius–Scavia of the Grothendieck–Serre conjecture for arbitrary smooth group schemes, and recent results of Harpaz, Morgan, and Skorobogatov on the Hasse principle for Kummer surfaces over number fields.

1. Integral Tate conjecture for 1-cycles over finite fields and a local–global principle for 0-cycles over global function fields

1.1. The Brauer-Manin obstruction for zero-cycles over global fields.

Conjecture 1.1 (Colliot-Thélène, Sansuc, Kato–Saito). *Let k be a global field, let X be a smooth projective geometrically connected variety over k, and let ℓ be a prime invertible in k. Define*

$$\widehat{\mathrm{CH}}_0(X) := \varprojlim_n \mathrm{CH}_0(X)/\ell^n.$$

For every place v of k, let k_v be the completion of k at v, let $\overline{k}_v$ be an algebraic closure of k_v, and define

$$\widehat{\mathrm{CH}}_0(X_{k_v}) := \begin{cases} \varprojlim_n \mathrm{CH}_0(X_{k_v})/\ell^n & \text{if } v \text{ is finite,} \\ \mathrm{CH}_0(X_{k_v})/N_{\overline{k}_v/k_v}(\mathrm{CH}_0(X_{\overline{k}_v}))[\ell] & \text{if } v \text{ is infinite.} \end{cases}$$

Then the following complex is exact:

$$\widehat{\mathrm{CH}}_0(X) \longrightarrow \prod_v \widehat{\mathrm{CH}}_0(X_{k_v}) \longrightarrow \mathrm{Hom}(\mathrm{Br}(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

Here, the product is over all places v of k, and the second map is induced by the pairing

$$\mathrm{CH}_0(X_{k_v}) \times \mathrm{Br}(X_{k_v}) \longrightarrow \mathrm{Br}(k_v) \xhookrightarrow{\mathrm{Inv}} \mathbb{Q}/\mathbb{Z},$$

where Inv *is the local invariant map.*

Recently, Tian [Tia25] proved the following important case of Conjecture 1.1.

Theorem 1.2 (Tian). *Let $\mathbb{F}$ be a finite field, let ℓ be a prime invertible in $\mathbb{F}$, let B be a smooth projective geometrically connected curve over $\mathbb{F}$, and let $X/\mathbb{F}(B)$ be a smooth projective surface such that $X_{\overline{\mathbb{F}(B)}}$ is rational. Then Conjecture 1.1 holds for X.*

Date: Mumbai, January 12 2026.

Corollary 1.3 (Tian)**.** *Assume that* $\mathrm{char}(\mathbb{F}) \neq 2$*. Let* X *be a del Pezzo surface of degree* 4 *over* $\mathbb{F}(B)$*, that is, a smooth complete intersection of two quadrics in* $\mathbb{P}^4_{\mathbb{F}(B)}$*. Then the Brauer–Manin obstruction to the Hasse principle for* X *is the only one.*

Proof. Since a del Pezzo surface is geometrically rational, by Theorem 1.2 (applied to $\ell = 2$) the existence of local points compatible with the Brauer–Manin pairing implies the existence of a zero-cycle of odd degree. By a result of Coray (later generalized by Brumer), this implies that X has a rational point. □

1.2. **Integral Tate conjecture for 1-cycles.**

Conjecture 1.4 (Integral Tate conjecture for 1-cycles)**.** *Let* $\mathbb{F}$ *be a finite field, and let* V *be a smooth projective geometrically connected variety of dimension* d *over* $\mathbb{F}$*. Let* ℓ *be a prime invertible in* $\mathbb{F}$*. Then the cycle class map*

$$\mathrm{CH}^{d-1}(V) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \xrightarrow{\mathrm{cl}} H^{2d-2}(V, \mathbb{Z}_\ell(d-1))$$

is surjective.

Remark 1.5. (1) If the ℓ-primary torsion of $\mathrm{Br}(V)$ is finite, then the cokernel of the cycle map is finite.

(2) By the Lefschetz hyperplane theorem, in order to prove Conjecture 1.4, it suffices to treat the case $d = 3$.

(3) Conjecture 1.4 was proved by Parimala and Suresh [PS16] when V is a conic bundle over a surface over $\mathbb{F}$.

Let $\mathcal{X}$ and B be smooth projective geometrically connected varieties over $\mathbb{F}$ of dimensions 3 and 1, respectively, let $f\colon \mathcal{X} \to B$ be a flat proper morphism, and let $X \coloneqq \mathcal{X} \times_B \mathbb{F}(B)$. Saito [Sai89] showed that Conjecture 1.4 for $\mathcal{X}$ implies a weak form of 1.1 for X. When X is a geometrically rational surface, one can show that Conjecture 1.4 for $\mathcal{X}$ implies the full 1.1 for X. Thus Theorem 1.2 is a consequence of the following theorem, which is the main contribution of [Tia25].

Theorem 1.6 (Tian)**.** *Let* $\mathbb{F}$ *be a finite field. Let* $\mathcal{X}$ *and* B *be smooth projective geometrically connected varieties over* $\mathbb{F}$ *of dimensions* 3 *and* 1*, respectively. Let* $f\colon \mathcal{X} \to B$ *be a flat proper morphism such that the generic fiber* $\mathcal{X}_{\mathbb{F}(B)}$ *is a geometrically rational surface. Then Conjecture 1.4 holds for* $\mathcal{X}$*.*

1.3. **Proof strategy for Theorem 1.6.**

Let $G \coloneqq \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ and let $\overline{\mathcal{X}} \coloneqq \mathcal{X} \times_{\mathbb{F}} \overline{\mathbb{F}}$. We consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(\overline{\mathrm{cl}}) & \longrightarrow & \mathrm{CH}^2(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell & & & & \\
 & & \downarrow{\scriptstyle \mathrm{AJ}} & & \downarrow{\scriptstyle \mathrm{cl}} & \searrow{\scriptstyle \overline{\mathrm{cl}}} & & & \\
0 & \longrightarrow & H^1(F, H^3(\overline{\mathcal{X}}, \mathbb{Z}_\ell(2))) & \longrightarrow & H^4(\mathcal{X}, \mathbb{Z}_\ell(2)) & \longrightarrow & H^4(\overline{\mathcal{X}}, \mathbb{Z}_\ell(2))^G & \longrightarrow & 0,
\end{array}$$

where AJ denotes the algebraic Abel–Jacobi map, cl is the cycle class map in ℓ-adic cohomology, and $\overline{\mathrm{cl}}$ the geometric cycle class map. Thus, it suffices to prove the two following assertions.

- The composite $\overline{\mathrm{cl}}$ is surjective: this is a corollary of the work of Kollár–Tian [KT25] which shows that algebraic equivalence of images of stable maps of curves lifts to deformation equivalence of the stable maps.

- The Abel–Jacobi map AJ is surjective: this is achieved in [Tia25] using the coniveau and strong coniveau filtrations (including results of [SS25]), Lawson homology via Voevodsky Chow sheaves, and also results from [KT25].

2. The Grothendieck–Serre conjecture for algebraic groups

2.1. **The theorem.** As first observed by Serre in the 1950s, the correct notion of a principal homogeneous space, or torsor, under a smooth group scheme G over a field k requires local triviality for the étale topology rather than for the Zariski topology. A natural question then arises: when is an étale G-torsor Zariski locally trivial? The following result gives a general answer.

Theorem 2.1. *Let X be a smooth scheme over a field k, and let G be a smooth group scheme over k. Then every generically trivial G-torsor over X is Zariski semi-locally trivial.*

Here, by definition, a G-torsor over X is said to be generically trivial if it is trivial over the generic point of every irreducible component of X.

2.2. **History of the problem.** The question of the validity of Theorem 2.1 was first raised by Serre in the Séminaire Chevalley of 1958 [Ser58], in the case where k is algebraically closed. Serre also proved the result when $G = \mathrm{PGL}_n$, as well as when G is an abelian variety. Colliot-Thélène and Ojanguren [CTO92] proved Theorem 2.1 when k is separably closed, and also when k is infinite perfect and G is reductive and totally isotropic. Raghunathan [Rag94, Rag95] established the result for infinite fields k and reductive groups G. The case when k is finite was treated by Gabber in unpublished work in the 1990s.

The general case of Theorem 2.1 has recently been proved in [BCS25]. In fact, smoothness of G is not needed: Theorem 2.1 holds for fppf torsors under a k-group G which satisfies the following condition, due to Gabber: Every $\overline{k}$-torus of $G_{\overline{k}}$ is contained in $(G^{\mathrm{gred}})_{\overline{k}}$, where $G^{\mathrm{gred}} \leq G$ denotes the maximal smooth k-subgroup of G.

In the Séminaire Chevalley of 1958 [Gro58], Grothendieck conjectured that for a reductive group scheme $\mathcal{G}$ over X, every generically trivial $\mathcal{G}$-torsor is Zariski-locally trivial. This conjecture was proved by Fedorov and Panin [FP15, Pan20]. Contrary to Serre's question, Grothendieck's conjecture fails beyond the case of reductive $\mathcal{G}$, as demonstrated by numerous examples [CTS87, BCS25].

2.3. **Strategy of proof.** The original proof strategy of Colliot-Thélène and Ojanguren uses a geometric presentation lemma due to Ojanguren to reduce the problem to the case when $X = \mathbb{A}^n_k$. They then complete the proof by an argument involving Raghunathan's rigidity theorem [Rag78] and the theorem of Raghunathan–Ramanathan's on the classification of G-torsors over $\mathbb{A}^1_k$ [RR84].

The approach used in [BCS25] follows the strategy of Fedorov–Panin, which is based on a presentation lemma of Gabber–Quillen (introduced by Quillen in his proof of the Gersten conjecture for algebraic K-theory, then refined by Gabber). For a regular semilocal k-algebra R and a generically trivial G-torsor E over R, Fedorov and Panin produce a G-torsor $\mathcal{E} \to \mathbb{P}^1_R$ which is trivial at $t = \infty$ and restricts to E at $t = 0$, where t is a coordinate on $\mathbb{P}^1_R$. The main goal of [BCS25] is to prove that every G-torsor over $\mathbb{P}^1_R$ which is trivial at $t = \infty$ is also trivial at $t = 0$: this is achieved by proving, for arbitrary smooth affine connected k-group

G containing no split unipotent k-group, purity theorems and extension theorems for G-torsors, a classification theorem for G-torsors over $\mathbb{P}^1_k$, and (under further assumptions on G) an unramifiedness property for the Whitehead group of G.

3. Schinzel's hypothesis (H) and applications to local–global principles

3.1. Schinzel's hypothesis (H).

Schinzel's hypothesis (H), formulated by Schinzel in 1958, is one of the most famous open problems in number theory. It is a very broad generalization of widely open conjectures such as the twin prime conjecture.

Conjecture 3.1 (Schinzel's hypothesis (H))**.** *Let $p_1(x), \dots, p_r(x) \in \mathbb{Z}[x]$ be irreducible polynomials. Assume:*

(1) $\gcd\{ p_1(m) \cdots p_r(m) \mid m \in \mathbb{Z} \} = 1$*, and*
(2) each p_i has positive leading coefficient.

Then there exist infinitely many integers n such that $p_i(n)$ is prime for all $1 \le i \le r$.

When $r = 1$ and $p_1(x) = ax + b$ with $\gcd(a, b) = 1$, Schinzel's hypothesis (H) reduces to Dirichlet's theorem on primes in arithmetic progressions. This is the only known case of Schinzel's hypothesis (H).

In 1924, Hasse used Dirichlet's theorem to prove the local-global principle for quadratic forms in four variables, starting from the corresponding result in three variables. Inspired by Hasse's argument, Colliot-Thélène and Sansuc [CTS82] proved the following conditional result.

Theorem 3.2 (Colliot-Thélène–Sansuc)**.** *Let $p(x) \in \mathbb{Q}[x]$ be an irreducible polynomial, and let $a \in \mathbb{Q}^\times$. Assume Schinzel's hypothesis* (H)*. Then the Hasse principle holds for any smooth proper model of the affine variety*

$$\{ y^2 - az^2 = p(x) \neq 0 \} \subset \mathbb{A}^3_{\mathbb{Q}}.$$

This result initiated a long series of statements of the form "Schinzel implies Hasse."

3.2. Swinnerton-Dyer's method.

Recall that the Tate–Shafarevich conjecture predicts that the Tate–Shafarevich group of any elliptic curve over a number field is finite. In [SD95], Swinnerton-Dyer introduced a method to prove results of the following form: assuming Schinzel's Hypothesis (H) and the Tate–Shafarevich conjecture, the Hasse principle holds for certain genus 1 fibrations $X \to \mathbb{P}^1_{\mathbb{Q}}$. He treated the case where X is a smooth intersection of two diagonal quadrics in $\mathbb{P}^3_{\mathbb{Q}}$, that is, a del Pezzo surface of degree 4, whose coefficients satisfy suitable arithmetic conditions.

We recall the basic idea of Swinnerton-Dyer's method. Let t denote a parameter on $\mathbb{P}^1_{\mathbb{Q}}$. Under some local arithmetic conditions, assuming X has points everywhere locally, Swinnerton-Dyer uses the fibration method to find $t \in \mathbb{Q}$ such that X_t has points everywhere locally. Let E_t be the Jacobian of the fiber X_t, so that X_t defines an element in the Tate–Shafarevich group $Ш^1(E_t)$. In the situations considered by Swinnerton-Dyer, the pencil $X \to \mathbb{P}^1_{\mathbb{Q}}$ has a multisection of degree 2, so that X_t defines an element of the 2-torsion subgroup $Ш^1(E_t)[2]$. We have the descent sequence

$$0 \longrightarrow E_t(\mathbb{Q})/2 \longrightarrow \mathrm{Sel}^{(2)}(E_t) \longrightarrow Ш^1(E_t)[2] \to 0,$$

where $\mathrm{Sel}^{(2)}(E_t)$ is the 2-Selmer group of E_t. In Swinnerton-Dyer's examples, one has $E_t(\mathbb{Q})/2 \simeq (\mathbb{Z}/2\mathbb{Z})^2$. By choosing $t \in \mathbb{Q}$ appropriately (this is the hardest part of the argument), Swinnerton-Dyer can ensure that $|\mathrm{Sel}^{(2)}(E_t)| \leq 8$, which implies $|\text{Ш}^1(E_t)[2]| \leq 2$. Under the Tate–Shafarevich conjecture, it is known by a theorem of Cassels that the order of $\text{Ш}(E_t)[2]$ is a square, so that $\text{Ш}^1(E_t)[2] = 0$. In particular, X_t has a rational point, and hence so does X, as desired.

3.3. **Refinements.** The method of Swinnerton-Dyer was later refined by Colliot-Thélène, Skorobogatov, and Swinnerton-Dyer [CTSSD98], who proved that, assuming Schinzel's Hypothesis (H) and the Tate–Shafarevich conjecture, the Brauer–Manin obstruction is the only obstruction to the Hasse principle for many genus 1 fibrations, including certain $K3$ surfaces. Building further on this approach, Wittenberg [Wit07] proved the analogous result for many del Pezzo surfaces of degree 4. The analogue of Wittenberg's theorem over global function fields is now unconditional, by the work of Tian and Kollár-Tian discussed at the beginning of this lecture.

More recently, Harpaz–Skorobogatov [HS16], Morgan–Skorobogatov [MS24] and Morgan [Mor25] proved that the Tate–Shafarevich conjecture implies the Hasse principle for certain Kummer surfaces, under certain algebraic and arithmetic hypotheses (which presumably imply the vanishing of the Brauer–Manin obstruction). In their situation, there are at most two degenerate fibers, so that Schinzel's hypothesis (H) is not needed. The main innovation is the replacement of Swinnerton-Dyer's difficult method to find t by arguments based on the study of Galois representations associated with 2-torsion points of abelian varieties. These arguments are inspired by the work of Mazur–Rubin [MR07].

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